Trigonometry
Solving Trig Equations – Part 1

Trig Identities, like the one to the right, are true for ALL angles: $\sin^2 x + \cos^2 x = 1$

Trig Equations, like the one to the right, are only true for SOME angles: $\sin x = 1$

When we ***solve*** trig equations, we must find **the values of x** that make the statement true. The equation, $\sin x = 1$, is true when $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ A general solution would be $x = \frac{\pi}{2} + 2\pi k$.

<u>Example</u>: Solve $2\sin x - 1 = 0$.

$$\left.\begin{aligned} 2\sin x - 1 &= 0 \\ 2\sin x &= 1 \\ \sin x &= \frac{1}{2} \end{aligned}\right\} \text{Isolate the } \sin x \text{ term}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right) \quad \rightarrow \text{Take the inverse sin}$$

$$\boxed{x = 30° + 360k,\ 150° + 360k}$$

→ Write an expression that lists all of the solutions, including coterminal angles.

<u>Example</u>: Solve $\sin x + \sqrt{2} = -\sin x$.

$$\sin x + \sqrt{2} = -\sin x$$

$$2\sin x + \sqrt{2} = 0 \quad \rightarrow \text{Collect } \sin x \text{ terms on one side.}$$

$$\left.\begin{aligned} 2\sin x &= -\sqrt{2} \\ \sin x &= -\frac{\sqrt{2}}{2} \end{aligned}\right\} \text{Isolate the } \sin x \text{ term}$$

$$x = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) \quad \rightarrow \text{Take the inverse sin}$$

$$\boxed{x = \frac{5\pi}{4} + 2\pi k,\ \frac{7\pi}{4} + 2\pi k}$$

Example: Solve $3\tan^2 x - 1 = 0$.

$$\left.\begin{aligned} 3\tan^2 x - 1 &= 0 \\ 3\tan^2 x &= 1 \\ \tan^2 x &= \frac{1}{3} \end{aligned}\right\} \text{ Isolate the } \tan^2 x \text{ term}$$

$$\sqrt{\tan^2 x} = \sqrt{\frac{1}{3}}$$ → Square root both sides

$$\tan x = \pm\frac{\sqrt{3}}{3}$$ → Don't forget the plus/minus

$$\boxed{x = 30° + 180°k,\ 150° + 180°k}$$

Example: Solve $2\cos 3x - 1 = 0$.

$$\left.\begin{aligned} 2\cos 3x - 1 &= 0 \\ 2\cos 3x &= 1 \\ \cos 3x &= \frac{1}{2} \end{aligned}\right\}$$

When the equation involves a *multiple angle*, don't divide by the coeffient of the angle, isolate the $\cos 3x$ term

$$3x = \cos^{-1}\frac{1}{2}$$

$$3x = 60° + 360k,\ \ 300° + 360k$$

$$\boxed{x = 20° + 120k,\ \ 100° + 120k}$$

→ Divide both sides by the coefficient of the angle.

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Sometimes the trig equation is in the form of a quadratic equation, and can be solved by either factoring or by using the quadratic formula.

<u>Example</u>: Solve $2\sin^2 x - \sin x - 1 = 0$.

$$2\sin^2 x - \sin x - 1 = 0$$
$$(\sin x - 1)(2\sin x + 1) = 0$$
$$\sin x = 1 \quad \text{and} \quad 2\sin x = -1$$
$$x = \sin^{-1} 1 \qquad \sin x = -\frac{1}{2}$$
$$\boxed{x = 90° + 360k} \qquad x = \sin^{-1}\left(-\frac{1}{2}\right)$$
$$\boxed{x = 210° + 360k,\ 330° + 360k}$$

<u>Example</u>: Solve $\cot x \cos^2 x = 2\cot x$.

$$\cot x \cos^2 x = 2\cot x$$
$$\cot x \cos^2 x - 2\cot x = 0$$
$$\cot x(\cos^2 x - 2) = 0$$
$$\cot x = 0 \quad \text{and} \quad \cos^2 x = 2$$
$$x = \cot^{-1} 0 \qquad \sqrt{\cos^2 x} = \sqrt{2}$$
$$\boxed{x = 90° + 180k} \qquad \cos x = \pm\sqrt{2}$$
~~$x = \cos^{-1} \pm\sqrt{2}$~~

cos x can't equal √2

Example: Solve $2\tan x \sin x + 2\sin x = \tan x + 1$.

$$2\tan x \sin x + 2\sin x = \tan x + 1$$
$$2\tan x \sin x + 2\sin x - \tan x - 1 = 0$$
$$2\sin x(\tan x + 1) - 1(\tan x + 1) = 0$$ → Factor by grouping!
$$(\tan x + 1)(2\sin x - 1) = 0$$

$$\tan x = -1 \qquad 2\sin x = 1$$
$$x = \tan^{-1}(-1) \qquad \sin x = \frac{1}{2}$$
$$\boxed{x = 135° + 180k} \qquad x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\boxed{\begin{array}{l} x = 30° + 360k \\ x = 150° + 360k \end{array}}$$

Example: Solve $\sin^2 x - 3\sin x - 2 = 0$.

$$\sin^2 x - 3\sin x - 2 = 0$$
$$\sin x = \frac{3 \pm \sqrt{9 - 4(1)(-2)}}{2}$$

Since this is not factorable, we will use the quadratic formula, solving for sin *x* instead of *x*.

$$\sin x = \frac{3 \pm \sqrt{17}}{2}$$
$$x = \sin^{-1}\left(\frac{3 \pm \sqrt{17}}{2}\right)$$

These values are not standard values, check to see if they are within the valid limits (-1 to 1); $\frac{3 + \sqrt{17}}{2}$ is outside the limit, so exclude it from the answer

$$\boxed{x = \sin^{-1}\left(\frac{3 - \sqrt{17}}{2}\right)}$$

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Solving Trig Equations – Part 3

Sometimes we need to use an identity to rewrite the equation into a form we can solve. If the equation is in the form of a quadratic trinomial, it also helps to write all the trig expressions in terms of the same trig function.

Example: $\sin^2 x + \cos 2x - \cos x = 0$

$$\sin^2 x + \cos 2x - \cos x = 0$$

→ Replace cos 2*x* with single angles

$$\sin^2 x + \cos^2 x - \sin^2 x - \cos x = 0$$

→ The $\sin^2 x$'s cancel, leaving an equation entirely in terms of cos *x*

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0 \qquad \cos x = 1$$

$$x = \cos^{-1} 0 \qquad x = \cos^{-1} 1$$

$$\boxed{x = 90° + 180k \qquad x = 0° + 360k}$$

Example: $2\sin^2 x + 3\cos x - 3 = 0$

$$2\sin^2 x + 3\cos x - 3 = 0$$

→ Replace $\sin^2 x$ with $1 - \cos^2 x$

$$2\left(1 - \cos^2 x\right) + 3\cos x - 3 = 0$$

$$2 - 2\cos^2 x + 3\cos x - 3 = 0$$

$$2\cos^2 x - 3\cos x + 1 = 0$$

$$(2\cos x - 1)(\cos x - 1) = 0$$

$$\cos x = \frac{1}{2} \qquad \cos x = 1$$

$$\boxed{\begin{array}{ll} x = 60° + 360k, & x = 0° + 360k \\ x = 300° + 360k & \end{array}}$$

Example: $\sec^2 x - 2\tan x = 4$

$$1 + \tan^2 x - 2\tan x = 4$$

→ Replace $\sec^2 x$ with $1 + \tan^2 x$

$$\tan^2 x - 2\tan x - 3 = 0$$

$$(\tan x - 3)(\tan x + 1) = 0$$

$$\tan x = 3 \qquad \tan x = -1$$

$$x = \tan^{-1} 3 \qquad x = \tan^{-1}(-1)$$

$\tan^{-1} 3$ is not a standard angle, we just simplify the expression

$$\boxed{x = \tan^{-1} 3 + 180k \qquad x = 135° + 180k}$$

Example: $\sin x = \tan x$

$$\sin x = \tan x$$

$$\sin x - \frac{\sin x}{\cos x} = 0$$

→ Replace tan *x* with sin *x*/cos *x*. This expression still has 2 different trig functions, but it is still factorable. We don't always need to rewrite the equation in terms of a single trig function.

$$\sin x\left(1 - \frac{1}{\cos x}\right) = 0$$

$$\sin x\left(1 - \sec x\right) = 0$$

$$\sin x = 0 \qquad \sec x = 1$$

$$x = \sin^{-1} 0 \qquad x = \sec^{-1} 1$$

$$x = 0° + 180k \qquad x = 0° + 360k$$

$$\boxed{x = 0° + 180k}$$

→ Both factors have the same solution, we only need to write it once.

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Solving Trig Equations – Part 4

In the next example, none of our identities will help us get the equation into a form we can solve, so we will **square both sides of the equation**. NOTE: When we square both sides of an equation, we subject ourselves to the possibility of ***extraneous solutions***! We will need to check our solutions before moving on!

Example: $\cos x + 1 = \sin x$

$$\cos x + 1 = \sin x$$
$$(\cos x + 1)^2 = \sin^2 x$$
$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$$
$$2\cos^2 x + 2\cos x = 0$$
$$2\cos x(\cos x + 1) = 0$$
$$\cos x = 0 \qquad \cos x = -1$$
$$x = \cos^{-1} 0 \qquad x = \cos^{-1}(-1)$$
$$\boxed{x = 90° + 180k \qquad x = 180° + 360k}$$

Test both sets of solutions.

$$\cos 90° + 1 = \sin 90°$$
$$0 + 1 = 1 \quad \text{(yes)}$$
$$\cos 270° + 1 = \sin 270°$$
$$0 + 1 = -1 \quad \text{(no)}$$

$$\cos 180° + 1 = \sin 180°$$
$$-1 + 1 = 0 \quad \text{(yes)}$$

Final solution: $\boxed{x = 90° + 360k \qquad x = 180° + 360k}$