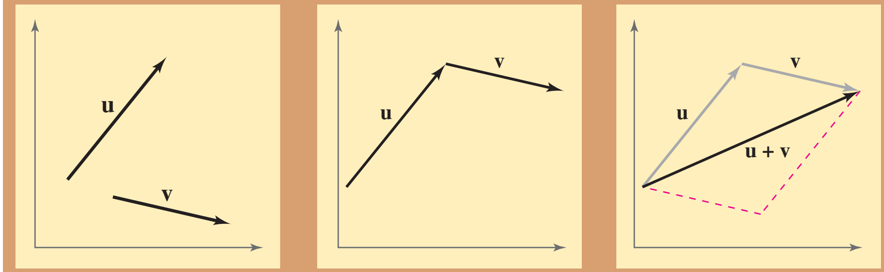
Chapter 6

Additional Topics in Trigonometry

Selected Applications

Triangles and vectors have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Flight Path, Exercise 27, page 415
- Bridge Design, Exercise 28, page 415
- Surveying, Exercise 38, page 422
- Landau Building, Exercise 45, page 422
- Velocity, Exercises 83 and 84, page 435
- Navigation, Exercises 89 and 90, page 436
- Revenue, Exercises 59 and 60, page 446
- Work, Exercise 63, page 447



Vectors indicate quantities that involve both magnitude and direction. In Chapter 6, you will study the operations of vectors in the plane and you will learn how to represent vector operations geometrically. You will also learn to solve oblique triangles and write complex numbers in trigonometric form.

Lester Lefkowitz/Getty Images

Vectors can be used to find the airspeed and direction of an airplane that will allow the airplane to maintain its groundspeed and direction.

6.1 Law of Sines

What you should learn

- Use the Law of Sines to solve oblique triangles (AAS or ASA).
- Use the Law of Sines to solve oblique triangles (SSA).
- Find areas of oblique triangles.
- Use the Law of Sines to model and solve real-life problems.

Why you should learn it

You can use the Law of Sines to solve real-life problems involving oblique triangles. For instance, Exercise 32 on page 415 shows how the Law of Sines can be used to help determine the distance from a boat to the shoreline.

Introduction

In Chapter 4 you looked at techniques for solving right triangles. In this section and the next, you will solve **oblique triangles**—triangles that have no right angles. As standard notation, the angles of a triangle are labeled A, B, and C, and their opposite sides are labeled a, b, and c, as shown in Figure 6.1.

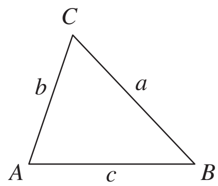


Figure 6.1

To solve an oblique triangle, you need to know the measure of at least one side and the measures of any two other parts of the triangle—two sides, two angles, or one angle and one side. This breaks down into the following four cases.

1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

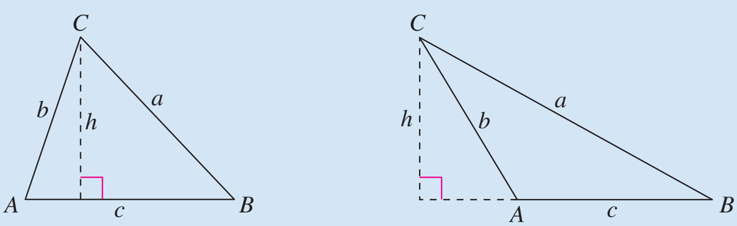
The first two cases can be solved using the **Law of Sines,** whereas the last two cases require the Law of Cosines (see Section 6.2).

Law of Sines (See the proof on page 468.)

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Oblique Triangles



A is acute. A is obtuse.

STUDY TIP

Notice in Figure 6.1 that angle A is the included angle between sides b and c, angle B is the included angle between sides a and c, and angle C is the included angle between sides a and b.

The Law of Sines can also be written in the reciprocal form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Example 1 Given Two Angles and One Side—AAS

For the triangle in Figure 6.2, $C = 102.3°$, $B = 28.7°$, and $b = 27.4$ feet. Find the remaining angle and sides.

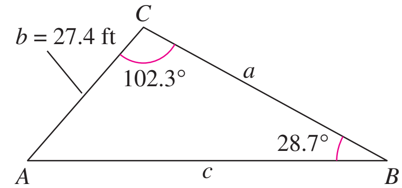


Figure 6.2

Solution

The third angle of the triangle is

$$\begin{aligned} A &= 180° - B - C \\ &= 180° - 28.7° - 102.3° \\ &= 49.0°. \end{aligned}$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Using $b = 27.4$ produces

$$a = \frac{b}{\sin B}(\sin A) = \frac{27.4}{\sin 28.7°}(\sin 49.0°) \approx 43.06 \text{ feet}$$

and

$$c = \frac{b}{\sin B}(\sin C) = \frac{27.4}{\sin 28.7°}(\sin 102.3°) \approx 55.75 \text{ feet.}$$

CHECKPOINT Now try Exercise 3.

Example 2 Given Two Angles and One Side—ASA

A pole tilts *toward* the sun at an 8° angle from the vertical, and it casts a 22-foot shadow. The angle of elevation from the tip of the shadow to the top of the pole is 43°. How tall is the pole?

Solution

In Figure 6.3, $A = 43°$ and $B = 90° + 8° = 98°$. So, the third angle is

$$C = 180° - A - B = 180° - 43° - 98° = 39°.$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Because $c = 22$ feet, the length of the pole is

$$a = \frac{c}{\sin C}(\sin A) = \frac{22}{\sin 39°}(\sin 43°) \approx 23.84 \text{ feet.}$$

CHECKPOINT Now try Exercise 25.

For practice, try reworking Example 2 for a pole that tilts *away from* the sun under the same conditions.

STUDY TIP

When you are solving triangles, a careful sketch is useful as a quick test for the feasibility of an answer. Remember that the longest side lies opposite the largest angle, and the shortest side lies opposite the smallest angle.

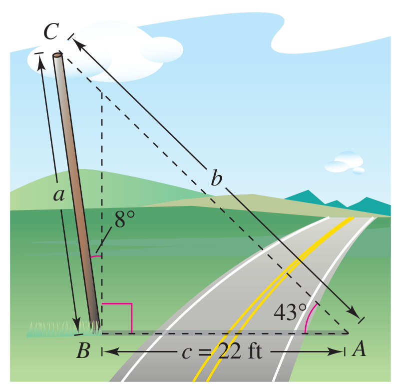


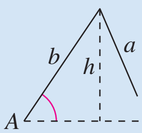
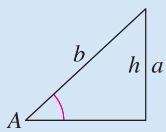
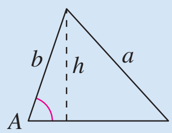
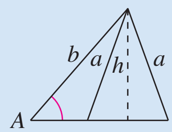
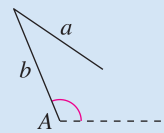
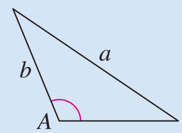
Figure 6.3

The Ambiguous Case (SSA)

In Examples 1 and 2 you saw that two angles and one side determine a unique triangle. However, if two sides and one opposite angle are given, three possible situations can occur: (1) no such triangle exists, (2) one such triangle exists, or (3) two distinct triangles satisfy the conditions.

The Ambiguous Case (SSA)

Consider a triangle in which you are given a, b, and A $(h = b \sin A)$.

	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Sketch						
Necessary condition	$a < h$	$a = h$	$a \ge b$	$h < a < b$	$a \le b$	$a > b$
Possible triangles	None	One	One	Two	None	One

Example 3 Single-Solution Case—SSA

For the triangle in Figure 6.4, $a = 22$ inches, $b = 12$ inches, and $A = 42°$. Find the remaining side and angles.

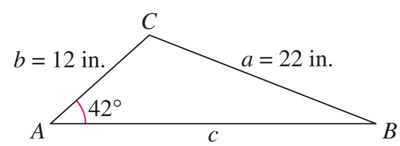


Figure 6.4 ***One solution:*** $a > b$

Solution

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \text{Reciprocal form}$$

$$\sin B = b\left(\frac{\sin A}{a}\right) \qquad \text{Multiply each side by } b.$$

$$\sin B = 12\left(\frac{\sin 42°}{22}\right) \qquad \text{Substitute for } A, a, \text{ and } b.$$

$$B \approx 21.41°. \qquad B \text{ is acute.}$$

Now you can determine that

$$C \approx 180° - 42° - 21.41° = 116.59°.$$

Then the remaining side is given by

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{22}{\sin 42°}(\sin 116.59°) \approx 29.40 \text{ inches.}$$

CHECKPOINT Now try Exercise 13.

Example 4 No-Solution Case—SSA

Show that there is no triangle for which $a = 15$, $b = 25$, and $A = 85°$.

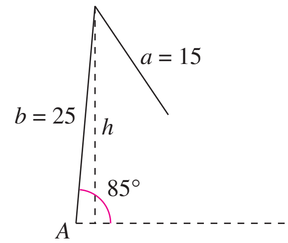


Figure 6.5 ***No solution:*** $a < h$

Solution

Begin by making the sketch shown in Figure 6.5. From this figure it appears that no triangle is formed. You can verify this by using the Law of Sines.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \text{Reciprocal form}$$

$$\sin B = b\left(\frac{\sin A}{a}\right) \qquad \text{Multiply each side by } b.$$

$$\sin B = 25\left(\frac{\sin 85°}{15}\right) \approx 1.6603 > 1$$

This contradicts the fact that $|\sin B| \leq 1$. So, no triangle can be formed having sides $a = 15$ and $b = 25$ and an angle of $A = 85°$.

CHECKPOINT Now try Exercise 15.

Example 5 Two-Solution Case—SSA

Find two triangles for which $a = 12$ meters, $b = 31$ meters, and $A = 20.5°$.

Solution

Because $h = b \sin A = 31(\sin 20.5°) \approx 10.86$ meters, you can conclude that there are two possible triangles (because $h < a < b$). By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \text{Reciprocal form}$$

$$\sin B = b\left(\frac{\sin A}{a}\right) = 31\left(\frac{\sin 20.5°}{12}\right) \approx 0.9047.$$

There are two angles $B_1 \approx 64.8°$ and $B_2 \approx 180° - 64.8° = 115.2°$ between 0° and 180° whose sine is 0.9047. For $B_1 \approx 64.8°$, you obtain

$$C \approx 180° - 20.5° - 64.8° = 94.7°$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{12}{\sin 20.5°}(\sin 94.7°) \approx 34.15 \text{ meters.}$$

For $B_2 \approx 115.2°$, you obtain

$$C \approx 180° - 20.5° - 115.2° = 44.3°$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{12}{\sin 20.5°}(\sin 44.3°) \approx 23.93 \text{ meters.}$$

The resulting triangles are shown in Figure 6.6.

CHECKPOINT Now try Exercise 17.

STUDY TIP

In Example 5, the height h of the triangle can be found using the formula

$$\sin A = \frac{h}{b}$$

or

$$h = b \sin A.$$

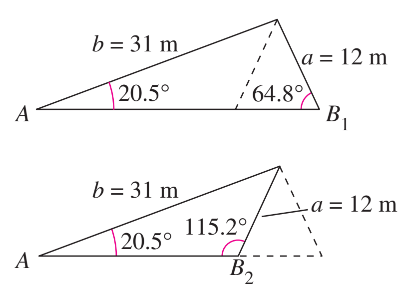


Figure 6.6 ***Two solutions:*** $h < a < b$

Area of an Oblique Triangle

The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle. Referring to Figure 6.7, note that each triangle has a height of $h = b \sin A$. To see this when A is obtuse, substitute the reference angle $180° - A$ for A. Now the height of the triangle is given by

$$h = b \sin(180° - A).$$

Using the difference formula for sine, the height is given by

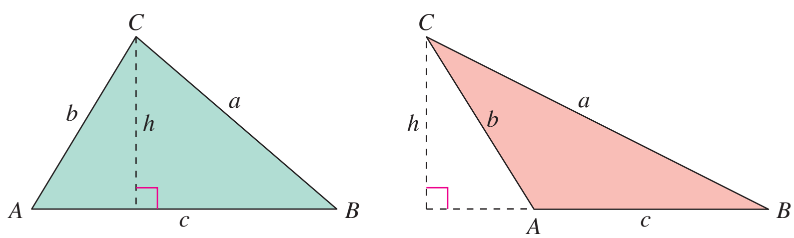
$$\begin{aligned} h &= b(\sin 180° \cos A - \cos 180° \sin A) && \sin(u - v) = \sin u \cos v - \cos u \sin v \\ &= b[0 \cdot \cos A - (-1) \cdot \sin A] \\ &= b \sin A. \end{aligned}$$

Consequently, the area of each triangle is given by

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2}bc \sin A.$$

By similar arguments, you can develop the formulas

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$



***A* is acute.** ***A* is obtuse.**

Figure 6.7

> **Area of an Oblique Triangle**
>
> The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,
>
> $$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Note that if angle A is $90°$, the formula gives the area of a right triangle as

$$\text{Area} = \frac{1}{2}bc = \frac{1}{2}(\text{base})(\text{height}).$$

Similar results are obtained for angles C and B equal to $90°$.

Example 6 Finding the Area of an Oblique Triangle

Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102°.

Solution

Consider $a = 90$ meters, $b = 52$ meters, and $C = 102°$, as shown in Figure 6.8. Then the area of the triangle is

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(90)(52)(\sin 102°) \approx 2288.87 \text{ square meters.}$$

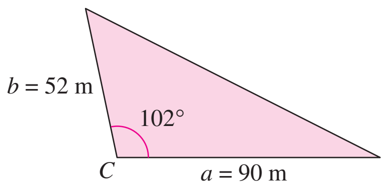


Figure 6.8

CHECKPOINT Now try Exercise 19.

Example 7 An Application of the Law of Sines

The course for a boat race starts at point A and proceeds in the direction S 52° W to point B, then in the direction S 40° E to point C, and finally back to A, as shown in Figure 6.9. Point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.

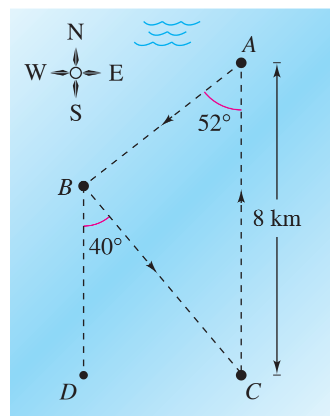


Figure 6.9

Solution

Because lines BD and AC are parallel, it follows that $\angle BCA \cong \angle DBC$. Consequently, triangle ABC has the measures shown in Figure 6.10. For angle B, you have $B = 180° - 52° - 40° = 88°$. Using the Law of Sines

$$\frac{a}{\sin 52°} = \frac{b}{\sin 88°} = \frac{c}{\sin 40°}$$

you can let $b = 8$ and obtain

$$a = \frac{8}{\sin 88°}(\sin 52°) \approx 6.31$$

and

$$c = \frac{8}{\sin 88°}(\sin 40°) \approx 5.15.$$

The total length of the course is approximately

$$\text{Length} \approx 8 + 6.31 + 5.15 = 19.46 \text{ kilometers.}$$

CHECKPOINT Now try Exercise 27.

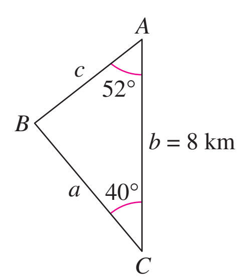


Figure 6.10

6.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. An _______ triangle is one that has no right angles.

2. Law of Sines: $\dfrac{a}{\sin A} = $ _______ $= \dfrac{c}{\sin C}$

3. The Law of Sines can be used to solve a triangle for cases

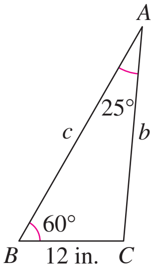
(a) _______ angle(s) and _______ side(s), which can be denoted _______ or _______ ,

(b) _______ side(s) and _______ angle(s), which can be denoted _______ .

4. To find the area of any triangle, use one of the following three formulas: Area = _______ , _______ , or _______ .

In Exercises 1–18, use the Law of Sines to solve the triangle. If two solutions exist, find both.

1.



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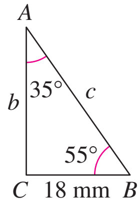

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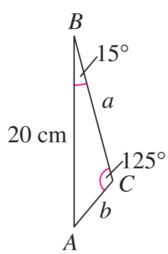

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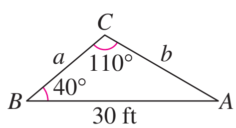

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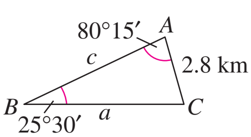

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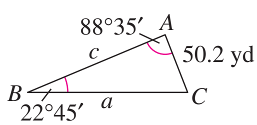


7. $A = 36°, \quad a = 8, \quad b = 5$

8. $A = 60°, \quad a = 9, \quad c = 10$

9. $A = 102.4°, \quad C = 16.7°, \quad a = 21.6$

10. $A = 24.3°, \quad C = 54.6°, \quad c = 2.68$

11. $A = 110° \, 15', \quad a = 48, \quad b = 16$

12. $B = 2° \, 45', \quad b = 6.2, \quad c = 5.8$

13. $A = 110°, \quad a = 125, \quad b = 100$

14. $A = 110°, \quad a = 125, \quad b = 200$

15. $A = 76°, \quad a = 18, \quad b = 20$

16. $A = 76°, \quad a = 34, \quad b = 21$

17. $A = 58°, \quad a = 11.4, \quad b = 12.8$

18. $A = 58°, \quad a = 4.5, \quad b = 12.8$

In Exercises 19–24, find the area of the triangle having the indicated angle and sides.

19. $C = 110°, \quad a = 6, \quad b = 10$

20. $B = 130°, \quad a = 92, \quad c = 30$

21. $A = 38° \, 45', \quad b = 67, \quad c = 85$

22. $A = 5° \, 15', \quad b = 4.5, \quad c = 22$

23. $B = 75° \, 15', \quad a = 103, \quad c = 58$

24. $C = 85° \, 45', \quad a = 16, \quad b = 20$

25. ***Height*** A flagpole at a right angle to the horizontal is located on a slope that makes an angle of 14° with the horizontal. The flagpole casts a 16-meter shadow up the slope when the angle of elevation from the tip of the shadow to the sun is 20°.

(a) Draw a triangle that represents the problem. Show the known quantities on the triangle and use a variable to indicate the height of the flagpole.

(b) Write an equation involving the unknown quantity.

(c) Find the height of the flagpole.

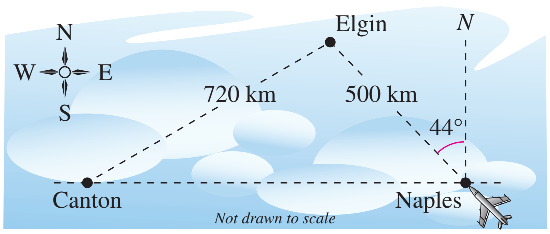
26. ***Height*** You are standing 40 meters from the base of a tree that is leaning 8° from the vertical away from you. The angle of elevation from your feet to the top of the tree is 20° 50′.

(a) Draw a triangle that represents the problem. Show the known quantities on the triangle and use a variable to indicate the height of the tree.

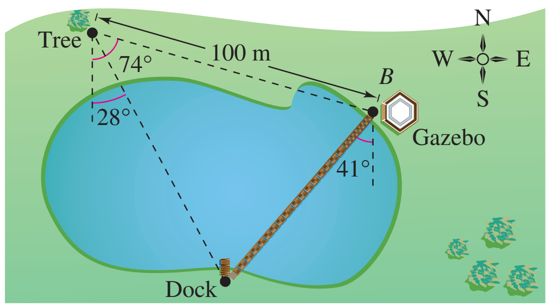
(b) Write an equation involving the unknown height of the tree.

(c) Find the height of the tree.

27. ***Flight Path*** A plane flies 500 kilometers with a bearing of 316° (clockwise from north) from Naples to Elgin (see figure). The plane then flies 720 kilometers from Elgin to Canton. Find the bearing of the flight from Elgin to Canton.



28. ***Bridge Design*** A bridge is to be built across a small lake from a gazebo to a dock (see figure). The bearing from the gazebo to the dock is S 41° W. From a tree 100 meters from the gazebo, the bearings to the gazebo and the dock are S 74° E and S 28° E, respectively. Find the distance from the gazebo to the dock.



29. ***Railroad Track Design*** The circular arc of a railroad curve has a chord of length 3000 feet and a central angle of 40°.

(a) Draw a diagram that visually represents the problem. Show the known quantities on the diagram and use the variables r and s to represent the radius of the arc and the length of the arc, respectively.

(b) Find the radius r of the circular arc.

(c) Find the length s of the circular arc.

30. ***Glide Path*** A pilot has just started on the glide path for landing at an airport with a runway of length 9000 feet. The angles of depression from the plane to the ends of the runway are 17.5° and 18.8°.

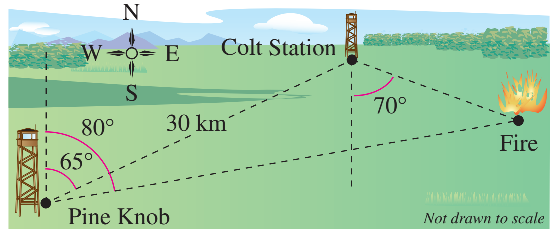
(a) Draw a diagram that visually represents the problem.

(b) Find the air distance the plane must travel until touching down on the near end of the runway.

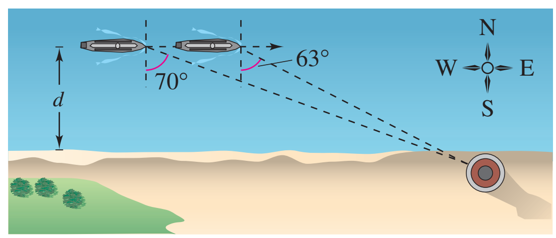
(c) Find the ground distance the plane must travel until touching down.

(d) Find the altitude of the plane when the pilot begins the descent.

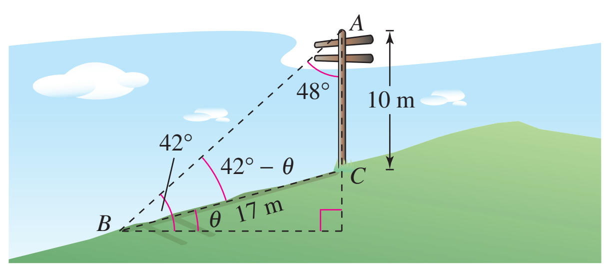
31. ***Locating a Fire*** The bearing from the Pine Knob fire tower to the Colt Station fire tower is N 65° E, and the two towers are 30 kilometers apart. A fire spotted by rangers in each tower has a bearing of N 80° E from Pine Knob and S 70° E from Colt Station. Find the distance of the fire from each tower.



32. ***Distance*** A boat is sailing due east parallel to the shoreline at a speed of 10 miles per hour. At a given time the bearing to a lighthouse is S 70° E, and 15 minutes later the bearing is S 63° E (see figure). The lighthouse is located at the shoreline. Find the distance from the boat to the shoreline.



33. ***Angle of Elevation*** A 10-meter telephone pole casts a 17-meter shadow directly down a slope when the angle of elevation of the sun is 42° (see figure). Find θ, the angle of elevation of the ground.

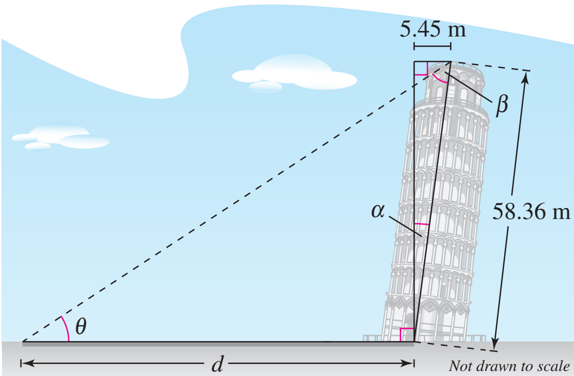


34. ***Distance*** The angles of elevation θ and ϕ to an airplane are being continuously monitored at two observation points A and B, respectively, which are 2 miles apart, and the airplane is east of both points in the same vertical plane.

(a) Draw a diagram that illustrates the problem.

(b) Write an equation giving the distance d between the plane and point B in terms of θ and ϕ.

35. ***Shadow Length*** The Leaning Tower of Pisa in Italy leans because it was built on unstable soil—a mixture of clay, sand, and water. The tower is approximately 58.36 meters tall from its foundation (see figure). The top of the tower leans about 5.45 meters off center.



(a) Find the angle of lean α of the tower.

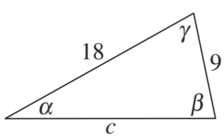
(b) Write β as a function of d and θ, where θ is the angle of elevation to the sun.

(c) Use the Law of Sines to write an equation for the length d of the shadow cast by the tower in terms of θ.

(d) Use a graphing utility to complete the table.

θ	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$
d						

36. ***Graphical and Numerical Analysis*** In the figure, α and β are positive angles.



(a) Write α as a function of β.

(b) Use a graphing utility to graph the function. Determine its domain and range.

(c) Use the result of part (b) to write c as a function of β.

(d) Use a graphing utility to graph the function in part (c). Determine its domain and range.

(e) Use a graphing utility to complete the table. What can you conclude?

β	0.4	0.8	1.2	1.6	2.0	2.4	2.8
α							
c							

Synthesis

True or False? **In Exercises 37 and 38, determine whether the statement is true or false. Justify your answer.**

37. If any three sides or angles of an oblique triangle are known, then the triangle can be solved.

38. If a triangle contains an obtuse angle, then it must be oblique.

39. ***Writing*** Can the Law of Sines be used to solve a right triangle? If so, write a short paragraph explaining how to use the Law of Sines to solve the following triangle. Is there an easier way to solve the triangle? Explain.

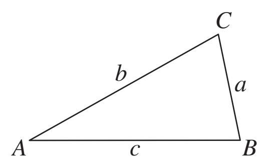
$B = 50°,\ C = 90°,\ a = 10$

40. ***Think About It*** Given $A = 36°$ and $a = 5$, find values of b such that the triangle has (a) one solution, (b) two solutions, and (c) no solution.

Mollweide's Formula **In Exercises 41 and 42, solve the triangle. Then use one of the two forms of Mollweide's Formula to verify the results.**

$$(a + b)\sin\left(\frac{C}{2}\right) = c\cos\left(\frac{A - B}{2}\right) \qquad \text{Form 1}$$

$$(a - b)\cos\left(\frac{C}{2}\right) = c\sin\left(\frac{A - B}{2}\right) \qquad \text{Form 2}$$



41. Solve the triangle for $A = 45°$, $B = 52°$, and $a = 16$. Then use form 1 of Mollweide's Formula to verify your solution.

42. Solve the triangle for $A = 42°$, $B = 60°$, and $a = 24$. Then use form 2 of Mollweide's Formula to verify your solution.

Skills Review

In Exercises 43 and 44, use the given values to find (if possible) the values of the remaining four trigonometric functions of θ.

43. $\cos\theta = \frac{5}{13},\ \sin\theta = -\frac{12}{13}$

44. $\tan\theta = \frac{2}{9},\ \csc\theta = -\frac{\sqrt{85}}{2}$

In Exercises 45–48, write the product as a sum or difference.

45. $6\sin 8\theta\cos 3\theta$

46. $2\cos 2\theta\cos 5\theta$

47. $3\cos\frac{\pi}{6}\sin\frac{5\pi}{3}$

48. $\frac{5}{2}\sin\frac{3\pi}{4}\sin\frac{5\pi}{6}$

6.2 Law of Cosines

Introduction

Two cases remain in the list of conditions needed to solve an oblique triangle—SSS and SAS. To use the Law of Sines, you must know at least one side and its opposite angle. If you are given three sides (SSS), or two sides and their included angle (SAS), none of the ratios in the Law of Sines would be complete. In such cases you can use the **Law of Cosines.**

Law of Cosines (See the proof on page 469.)

Standard Form	*Alternative Form*
$a^2 = b^2 + c^2 - 2bc \cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac \cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab \cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example 1 Three Sides of a Triangle—SSS

Find the three angles of the triangle shown in Figure 6.11.

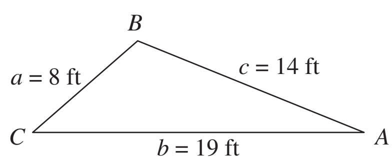


Figure 6.11

Solution

It is a good idea first to find the angle opposite the longest side—side b in this case. Using the alternative form of the Law of Cosines, you find that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 14^2 - 19^2}{2(8)(14)} \approx -0.45089.$$

Because $\cos B$ is negative, you know that B is an *obtuse* angle given by $B \approx 116.80°$. At this point it is simpler to use the Law of Sines to determine A.

$$\sin A = a\left(\frac{\sin B}{b}\right) \approx 8\left(\frac{\sin 116.80°}{19}\right) \approx 0.37583$$

Because B is obtuse, A must be acute, because a triangle can have at most one obtuse angle. So, $A \approx 22.08°$ and $C \approx 180° - 22.08° - 116.80° = 41.12°$.

CHECKPOINT Now try Exercise 1.

What you should learn

- Use the Law of Cosines to solve oblique triangles (SSS or SAS).
- Use the Law of Cosines to model and solve real-life problems.
- Use Heron's Area Formula to find areas of triangles.

Why you should learn it

You can use the Law of Cosines to solve real-life problems involving oblique triangles. For instance, Exercise 42 on page 422 shows you how the Law of Cosines can be used to determine the length of the guy wires that anchor a tower.

Do you see why it was wise to find the largest angle *first* in Example 1? Knowing the cosine of an angle, you can determine whether the angle is acute or obtuse. That is,

$\cos\theta > 0$ for $0° < \theta < 90°$ Acute

$\cos\theta < 0$ for $90° < \theta < 180°$. Obtuse

So, in Example 1, once you found that angle B was obtuse, you knew that angles A and C were both acute. Furthermore, if the largest angle is acute, the remaining two angles are also acute.

Exploration

What familiar formula do you obtain when you use the third form of the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

and you let $C = 90°$? What is the relationship between the Law of Cosines and this formula?

Example 2 Two Sides and the Included Angle—SAS

Find the remaining angles and side of the triangle shown in Figure 6.12.

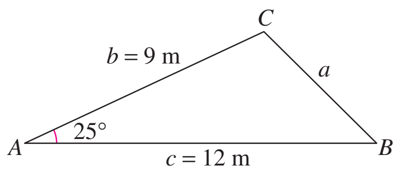


Figure 6.12

Solution

Use the Law of Cosines to find the unknown side a in the figure.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = 9^2 + 12^2 - 2(9)(12)\cos 25° \approx 29.2375$$

$$a \approx 5.4072$$

Because $a \approx 5.4072$ meters, you now know the ratio $\sin A/a$ and you can use the reciprocal form of the Law of Sines to solve for B.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = b\left(\frac{\sin A}{a}\right) = 9\left(\frac{\sin 25°}{5.4072}\right) \approx 0.7034$$

There are two angles between 0° and 180° whose sine is 0.7034, $B_1 \approx 44.7°$ and $B_2 \approx 180° - 44.7° = 135.3°$. For $B_1 \approx 44.7°$,

$$C_1 \approx 180° - 25° - 44.7° = 110.3°.$$

For $B_2 \approx 135.3°$,

$$C_2 \approx 180° - 25° - 135.3° = 19.7°.$$

Because side c is the longest side of the triangle, C must be the largest angle of the triangle. So, $B \approx 44.7°$ and $C \approx 110.3°$.

CHECKPOINT Now try Exercise 5.

STUDY TIP

When solving an oblique triangle given three sides, you use the alternative form of the Law of Cosines to solve for an angle. When solving an oblique triangle given two sides and their included angle, you use the standard form of the Law of Cosines to solve for an unknown side.

Exploration

In Example 2, suppose $A = 115°$. After solving for a, which angle would you solve for next, B or C? Are there two possible solutions for that angle? If so, how can you determine which angle is the correct solution?

Applications

Example 3 An Application of the Law of Cosines

The pitcher's mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet, as shown in Figure 6.13. (The pitcher's mound is *not* halfway between home plate and second base.) How far is the pitcher's mound from first base?

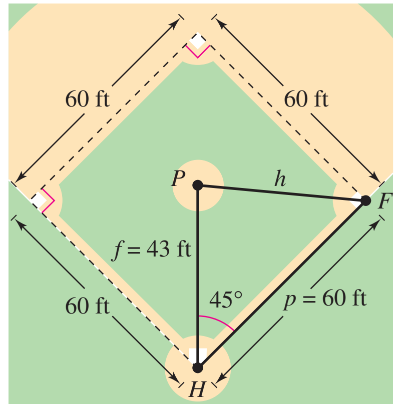


Figure 6.13

Solution

In triangle HPF, $H = 45°$ (line HP bisects the right angle at H), $f = 43$, and $p = 60$. Using the Law of Cosines for this SAS case, you have

$$h^2 = f^2 + p^2 - 2fp \cos H$$

$$= 43^2 + 60^2 - 2(43)(60) \cos 45°$$

$$\approx 1800.33.$$

So, the approximate distance from the pitcher's mound to first base is

$$h \approx \sqrt{1800.33} \approx 42.43 \text{ feet.}$$

CHECKPOINT Now try Exercise 37.

Example 4 An Application of the Law of Cosines

A ship travels 60 miles due east, then adjusts its course northward, as shown in Figure 6.14. After traveling 80 miles in the new direction, the ship is 139 miles from its point of departure. Describe the bearing from point B to point C.

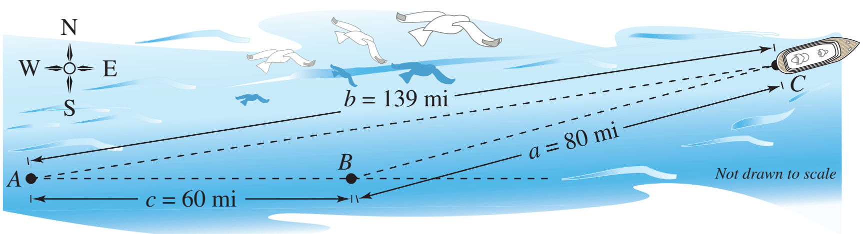


Figure 6.14

Solution

You have $a = 80$, $b = 139$, and $c = 60$; so, using the alternative form of the Law of Cosines, you have

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{80^2 + 60^2 - 139^2}{2(80)(60)} \approx -0.97094.$$

So, $B \approx \arccos(-0.97094) \approx 166.15°$. Therefore, the bearing measured from due north from point B to point C is $166.15° - 90° = 76.15°$, or N 76.15° E.

CHECKPOINT Now try Exercise 39.

Heron's Area Formula

The Law of Cosines can be used to establish the following formula for the area of a triangle. This formula is called **Heron's Area Formula** after the Greek mathematician Heron (ca. 100 B.C.).

Heron's Area Formula (See the proof on page 470.)

Given any triangle with sides of lengths a, b, and c, the area of the triangle is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$.

Example 5 Using Heron's Area Formula

Find the area of a triangle having sides of lengths $a = 43$ meters, $b = 53$ meters, and $c = 72$ meters.

Solution

Because $s = (a+b+c)/2 = 168/2 = 84$, Heron's Area Formula yields

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{84(84-43)(84-53)(84-72)}\\ &= \sqrt{84(41)(31)(12)}\\ &\approx 1131.89 \text{ square meters.}\end{aligned}$$

CHECKPOINT Now try Exercise 45.

You have now studied three different formulas for the area of a triangle.

Formulas for Area of a Triangle

1. Standard Formula: $\text{Area} = \frac{1}{2}bh$
2. Oblique Triangle: $\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$
3. Heron's Area Formula: $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

Exploration

Can the formulas above be used to find the area of any type of triangle? Explain the advantages and disadvantages of using one formula over another.

6.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

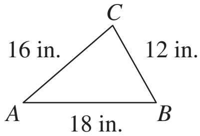
1. The standard form of the Law of Cosines for $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ is ________ .

2. ________ Formula is established by using the Law of Cosines.

3. Three different formulas for the area of a triangle are given by Area $=$ ________ , Area $= \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$, and Area $=$ ________ .

In Exercises 1–20, use the Law of Cosines to solve the triangle.

1.



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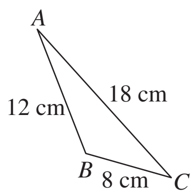

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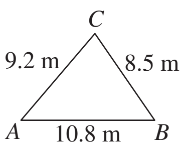

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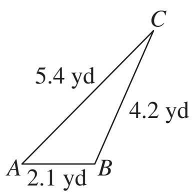

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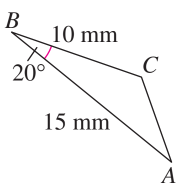

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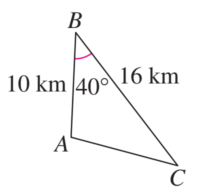

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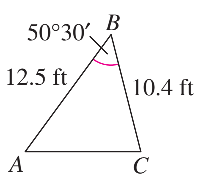

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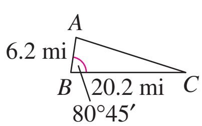


9. $a = 6, \quad b = 8, \quad c = 12$

10. $a = 9, \quad b = 3, \quad c = 11$

11. $A = 50°, \quad b = 15, \quad c = 30$

12. $C = 108°, \quad a = 10, \quad b = 7$

13. $a = 9, \quad b = 12, \quad c = 15$

14. $a = 45, \quad b = 30, \quad c = 72$

15. $a = 75.4, \quad b = 48, \quad c = 48$

16. $a = 1.42, \quad b = 0.75, \quad c = 1.25$

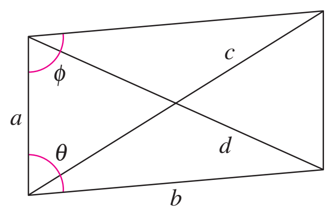
17. $B = 8° \, 15', \quad a = 26, \quad c = 18$

18. $B = 10° \, 35', \quad a = 40, \quad c = 30$

19. $B = 75° \, 20', \quad a = 6.2, \quad c = 9.5$

20. $C = 15° \, 15', \quad a = 6.25, \quad b = 2.15$

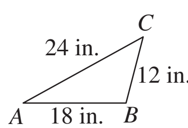
In Exercises 21–26, complete the table by solving the parallelogram shown in the figure. (The lengths of the diagonals are given by *c* and *d*.)



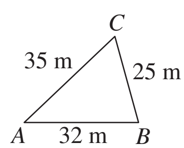
	a	b	c	d	θ	ϕ
21.	4	8			30°	
22.	25	35				120°
23.	10	14	20			
24.	40	60		80		
25.	15		25	20		
26.		25	50	35		

In Exercises 27–36, use Heron's Area Formula to find the area of the triangle.

27.



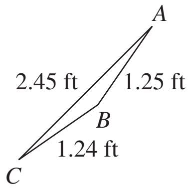
28.



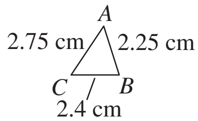
29. $a = 5, \quad b = 8, \quad c = 10$

30. $a = 14, \quad b = 17, \quad c = 7$

31.



32.



33. $a = 3.5, \quad b = 10.2, \quad c = 9$

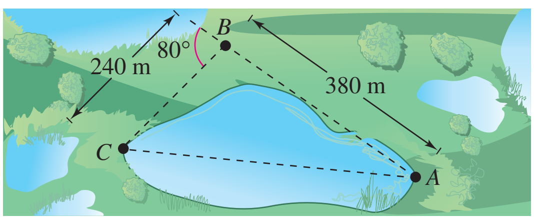
34. $a = 75.4, \quad b = 52, \quad c = 52$

35. $a = 10.59, \quad b = 6.65, \quad c = 12.31$

36. $a = 4.45, \quad b = 1.85, \quad c = 3.00$

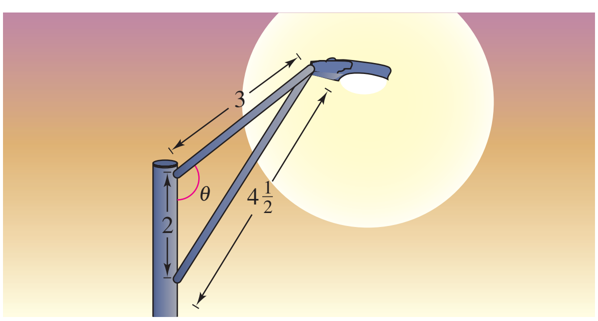
37. ***Navigation*** A plane flies 810 miles from Franklin to Centerville with a bearing of 75° (clockwise from north). Then it flies 648 miles from Centerville to Rosemont with a bearing of 32°. Draw a diagram that visually represents the problem, and find the straight-line distance and bearing from Rosemont to Franklin.

38. ***Surveying*** To approximate the length of a marsh, a surveyor walks 380 meters from point A to point B. Then the surveyor turns 80° and walks 240 meters to point C (see figure). Approximate the length AC of the marsh.



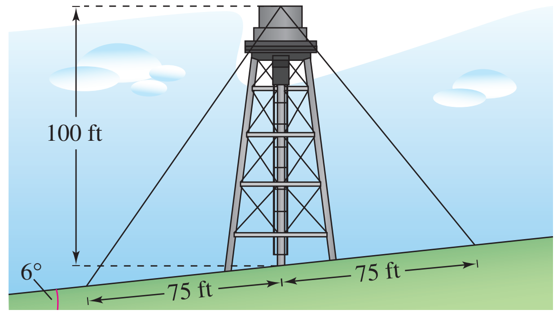
39. ***Navigation*** A boat race runs along a triangular course marked by buoys A, B, and C. The race starts with the boats headed west for 3600 meters. The other two sides of the course lie to the north of the first side, and their lengths are 1500 meters and 2800 meters. Draw a diagram that visually represents the problem, and find the bearings for the last two legs of the race.

40. ***Streetlight Design*** Determine the angle θ in the design of the streetlight shown in the figure.

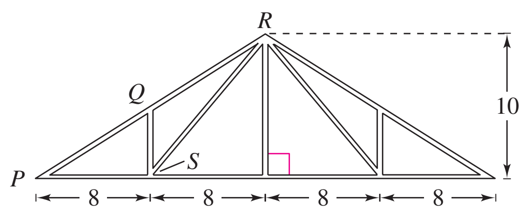


41. ***Distance*** Two ships leave a port at 9 A.M. One travels at a bearing of N 53° W at 12 miles per hour, and the other travels at a bearing of S 67° W at 16 miles per hour. Approximate how far apart the ships are at noon that day.

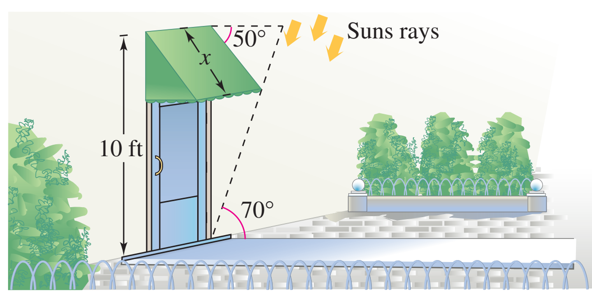
42. ***Length*** A 100-foot vertical tower is to be erected on the side of a hill that makes a 6° angle with the horizontal (see figure). Find the length of each of the two guy wires that will be anchored 75 feet uphill and downhill from the base of the tower.



43. ***Trusses*** Q is the midpoint of the line segment $\overline{PR}$ in the truss rafter shown in the figure. What are the lengths of the line segments $\overline{PQ}$, $\overline{QS}$, and $\overline{RS}$?

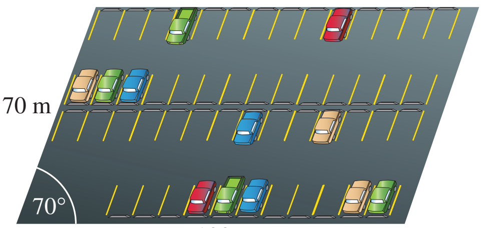


44. ***Awning Design*** A retractable awning above a patio lowers at an angle of 50° from the exterior wall at a height of 10 feet above the ground (see figure). No direct sunlight is to enter the door when the angle of elevation of the sun is greater than 70°. What is the length x of the awning?



45. ***Landau Building*** The Landau Building in Cambridge, Massachusetts has a triangular-shaped base. The lengths of the sides of the triangular base are 145 feet, 257 feet, and 290 feet. Find the area of the base of the building.

46. ***Geometry*** A parking lot has the shape of a parallelogram (see figure). The lengths of two adjacent sides are 70 meters and 100 meters. The angle between the two sides is 70°. What is the area of the parking lot?



47. ***Engine Design*** An engine has a seven-inch connecting rod fastened to a crank (see figure).

(a) Use the Law of Cosines to write an equation giving the relationship between x and θ.

(b) Write x as a function of θ. (Select the sign that yields positive values of x.)

(c) Use a graphing utility to graph the function in part (b).

(d) Use the graph in part (c) to determine the total distance the piston moves in one cycle.

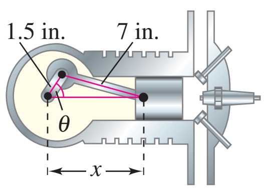


Figure for 47

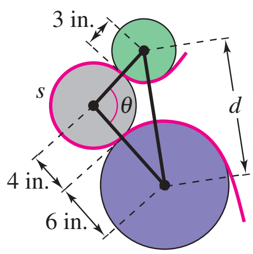


Figure for 48

48. ***Manufacturing*** In a process with continuous paper, the paper passes across three rollers of radii 3 inches, 4 inches, and 6 inches (see figure). The centers of the three-inch and six-inch rollers are d inches apart, and the length of the arc in contact with the paper on the four-inch roller is s inches.

(a) Use the Law of Cosines to write an equation giving the relationship between d and θ.

(b) Write θ as a function of d.

(c) Write s as a function of θ.

(d) Complete the table.

d (inches)	9	10	12	13	14	15	16
θ (degrees)							
s (inches)							

Synthesis

True or False? **In Exercises 49–51, determine whether the statement is true or false. Justify your answer.**

49. A triangle with side lengths of 10 feet, 16 feet, and 5 feet can be solved using the Law of Cosines.

50. Two sides and their included angle determine a unique triangle.

51. In Heron's Area Formula, s is the average of the lengths of the three sides of the triangle.

Proofs **In Exercises 52–54, use the Law of Cosines to prove each of the following.**

52. $\frac{1}{2}bc(1 + \cos A) = \left(\frac{a + b + c}{2}\right)\left(\frac{-a + b + c}{2}\right)$

53. $\frac{1}{2}bc(1 - \cos A) = \left(\frac{a - b + c}{2}\right)\left(\frac{a + b - c}{2}\right)$

54. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

55. ***Proof*** Use a half-angle formula and the Law of Cosines to show that, for any triangle,

$$\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s - c)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

56. ***Proof*** Use a half-angle formula and the Law of Cosines to show that, for any triangle,

$$\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s - a)(s - b)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

57. ***Writing*** Describe how the Law of Cosines can be used to solve the ambiguous case of the oblique triangle ABC, where $a = 12$ feet, $b = 30$ feet, and $A = 20°$. Is the result the same as when the Law of Sines is used to solve the triangle? Describe the advantages and the disadvantages of each method.

58. ***Writing*** In Exercise 57, the Law of Cosines was used to solve a triangle in the two-solution case of SSA. Can the Law of Cosines be used to solve the no-solution and single-solution cases of SSA? Explain.

Skills Review

In Exercises 59–62, evaluate the expression without using a calculator.

59. $\arcsin(-1)$

60. $\cos^{-1} 0$

61. $\tan^{-1} \sqrt{3}$

62. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

6.3 Vectors in the Plane

What you should learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent vectors graphically.
- Write vectors as linear combinations of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.

Why you should learn it

Vectors are used to analyze numerous aspects of everyday life. Exercise 86 on page 435 shows you how vectors can be used to determine the tension in the cables of two cranes lifting an object.

Sandra Baker/Getty Images

Introduction

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number. Other quantities, such as force and velocity, involve both *magnitude* and *direction* and cannot be completely characterized by a single real number. To represent such a quantity, you can use a **directed line segment,** as shown in Figure 6.15. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q. Its **magnitude,** or **length,** is denoted by $\|\overrightarrow{PQ}\|$ and can be found by using the Distance Formula.

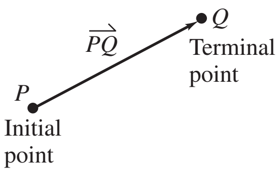


Figure 6.15

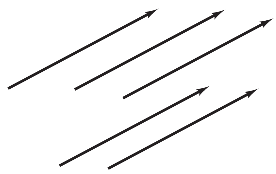


Figure 6.16

Two directed line segments that have the same magnitude and direction are *equivalent*. For example, the directed line segments in Figure 6.16 are all equivalent. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector v in the plane,** written $\mathbf{v} = \overrightarrow{PQ}$. Vectors are denoted by lowercase, boldface letters such as **u**, **v**, and **w**.

Example 1 Equivalent Directed Line Segments

Let **u** be represented by the directed line segment from $P = (0, 0)$ to $Q = (3, 2)$, and let **v** be represented by the directed line segment from $R = (1, 2)$ to $S = (4, 4)$, as shown in Figure 6.17. Show that $\mathbf{u} = \mathbf{v}$.

Solution

From the Distance Formula, it follows that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same magnitude*.

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction*, because they are both directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

So, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, and it follows that $\mathbf{u} = \mathbf{v}$.

CHECKPOINT Now try Exercise 1.

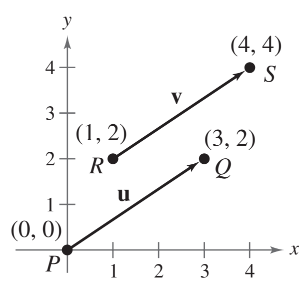


Figure 6.17

Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector **v** is in **standard position.**

A vector whose initial point is at the origin $(0, 0)$ can be uniquely represented by the coordinates of its terminal point (v_1, v_2). This is the **component form of a vector v,** written as

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are the *components* of **v**. If both the initial point and the terminal point lie at the origin, **v** is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Component Form of a Vector

The component form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of **v** is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are *equal* if and only if $u_1 = v_1$ and $u_2 = v_2$. For instance, in Example 1, the vector **u** from $P = (0, 0)$ to $Q = (3, 2)$ is

$$\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

and the vector **v** from $R = (1, 2)$ to $S = (4, 4)$ is

$$\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle.$$

TECHNOLOGY TIP

You can graph vectors with a graphing utility by graphing directed line segments. Consult the user's guide for your graphing utility for specific instructions.

Example 2 Finding the Component Form of a Vector

Find the component form and magnitude of the vector **v** that has initial point $(4, -7)$ and terminal point $(-1, 5)$.

Solution

Let $P = (4, -7) = (p_1, p_2)$ and $Q = (-1, 5) = (q_1, q_2)$, as shown in Figure 6.18. Then, the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$v_1 = q_1 - p_1 = -1 - 4 = -5$$

$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

So, $\mathbf{v} = \langle -5, 12 \rangle$ and the magnitude of **v** is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13.$$

CHECKPOINT Now try Exercise 5.

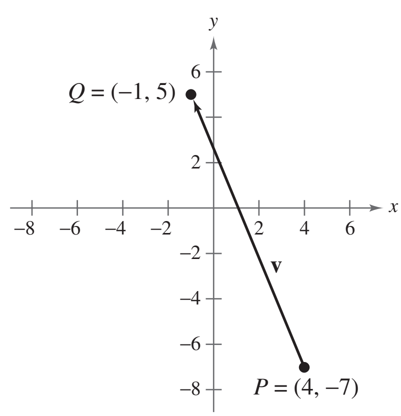


Figure 6.18

Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition.** Geometrically, the product of a vector $\mathbf{v}$ and a scalar k is the vector that is $|k|$ times as long as $\mathbf{v}$. If k is positive, $k\mathbf{v}$ has the same direction as $\mathbf{v}$, and if k is negative, $k\mathbf{v}$ has the opposite direction of $\mathbf{v}$, as shown in Figure 6.19.

To add two vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector $\mathbf{v}$ coincides with the terminal point of the first vector $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector formed by joining the initial point of the first vector $\mathbf{u}$ with the terminal point of the second vector $\mathbf{v}$, as shown in Figure 6.20. This technique is called the **parallelogram law** for vector addition because the vector $\mathbf{u} + \mathbf{v}$, often called the **resultant** of vector addition, is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.

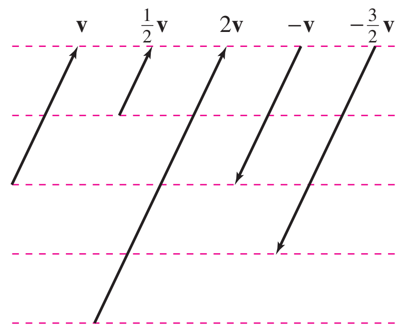


Figure 6.19

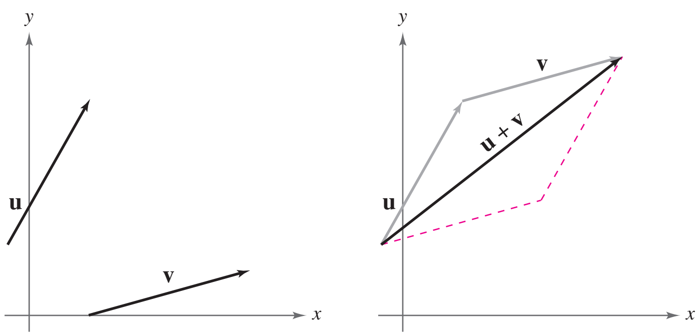


Figure 6.20

Definition of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Sum}$$

and the **scalar multiple** of k times $\mathbf{u}$ is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \qquad \text{Scalar multiple}$$

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\begin{aligned} -\mathbf{v} &= (-1)\mathbf{v} \\ &= \langle -v_1, -v_2 \rangle \qquad \text{Negative} \end{aligned}$$

and the **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} \mathbf{u} - \mathbf{v} &= \mathbf{u} + (-\mathbf{v}) \qquad \text{Add } (-\mathbf{v})\text{. See Figure 6.21.} \\ &= \langle u_1 - v_1, u_2 - v_2 \rangle. \qquad \text{Difference} \end{aligned}$$

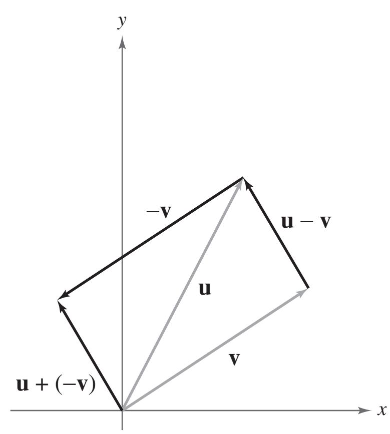


Figure 6.21

To represent $\mathbf{u} - \mathbf{v}$ geometrically, you can use directed line segments with the *same* initial point. The difference $\mathbf{u} - \mathbf{v}$ is the vector from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$, which is equal to $\mathbf{u} + (-\mathbf{v})$, as shown in Figure 6.21.

The component definitions of vector addition and scalar multiplication are illustrated in Example 3. In this example, notice that each of the vector operations can be interpreted geometrically.

Example 3 Vector Operations

Let $\mathbf{v} = \langle -2, 5\rangle$ and $\mathbf{w} = \langle 3, 4\rangle$, and find each of the following vectors.

a. $2\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$ **d.** $2\mathbf{v} - 3\mathbf{w}$

Solution

a. Because $\mathbf{v} = \langle -2, 5\rangle$, you have

$$\begin{aligned} 2\mathbf{v} &= 2\langle -2, 5\rangle \\ &= \langle 2(-2), 2(5)\rangle \\ &= \langle -4, 10\rangle. \end{aligned}$$

A sketch of $2\mathbf{v}$ is shown in Figure 6.22.

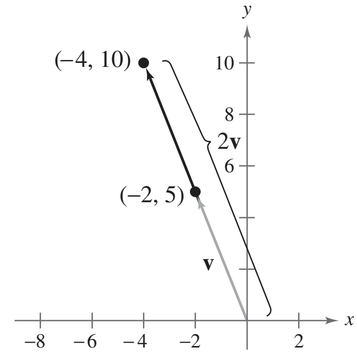


Figure 6.22

b. The difference of $\mathbf{w}$ and $\mathbf{v}$ is

$$\begin{aligned} \mathbf{w} - \mathbf{v} &= \langle 3 - (-2), 4 - 5\rangle \\ &= \langle 5, -1\rangle. \end{aligned}$$

A sketch of $\mathbf{w} - \mathbf{v}$ is shown in Figure 6.23. Note that the figure shows the vector difference $\mathbf{w} - \mathbf{v}$ as the sum $\mathbf{w} + (-\mathbf{v})$.

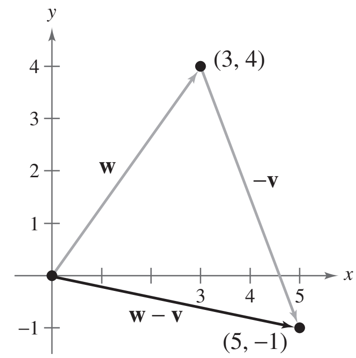


Figure 6.23

c. The sum of $\mathbf{v}$ and $2\mathbf{w}$ is

$$\begin{aligned} \mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + 2\langle 3, 4\rangle \\ &= \langle -2, 5\rangle + \langle 2(3), 2(4)\rangle \\ &= \langle -2, 5\rangle + \langle 6, 8\rangle \\ &= \langle -2 + 6, 5 + 8\rangle \\ &= \langle 4, 13\rangle. \end{aligned}$$

A sketch of $\mathbf{v} + 2\mathbf{w}$ is shown in Figure 6.24.

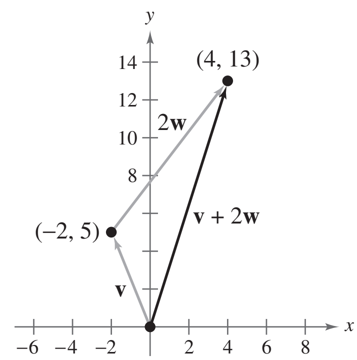


Figure 6.24

d. The difference of $2\mathbf{v}$ and $3\mathbf{w}$ is

$$\begin{aligned} 2\mathbf{v} - 3\mathbf{w} &= 2\langle -2, 5\rangle - 3\langle 3, 4\rangle \\ &= \langle 2(-2), 2(5)\rangle - \langle 3(3), 3(4)\rangle \\ &= \langle -4, 10\rangle - \langle 9, 12\rangle \\ &= \langle -4 - 9, 10 - 12\rangle \\ &= \langle -13, -2\rangle. \end{aligned}$$

A sketch of $2\mathbf{v} - 3\mathbf{w}$ is shown in Figure 6.25. Note that the figure shows the vector difference $2\mathbf{v} - 3\mathbf{w}$ as the sum $2\mathbf{v} + (-3\mathbf{w})$.

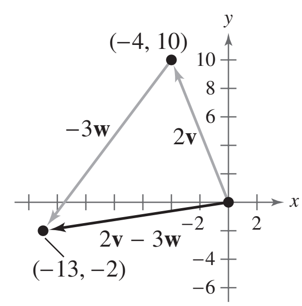


Figure 6.25

CHECKPOINT Now try Exercise 25.

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic.

Properties of Vector Addition and Scalar Multiplication

Let **u**, **v**, and **w** be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $c(d\mathbf{u}) = (cd)\mathbf{u}$

6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

8. $1(\mathbf{u}) = \mathbf{u},\ 0(\mathbf{u}) = \mathbf{0}$

9. $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$

STUDY TIP

Property 9 can be stated as follows: The magnitude of the vector $c\mathbf{v}$ is the absolute value of c times the magnitude of **v**.

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector **v**. To do this, you can divide **v** by its length to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}. \qquad \text{Unit vector in direction of } \mathbf{v}$$

Note that **u** is a scalar multiple of **v**. The vector **u** has a magnitude of 1 and the same direction as **v**. The vector **u** is called a **unit vector in the direction of v.**

Example 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that the result has a magnitude of 1.

Solution

The unit vector in the direction of **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}}$$

$$= \frac{1}{\sqrt{29}}\langle -2, 5\rangle$$

$$= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle = \left\langle \frac{-2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29} \right\rangle.$$

This vector has a magnitude of 1 because

$$\sqrt{\left(\frac{-2\sqrt{29}}{29}\right)^2 + \left(\frac{5\sqrt{29}}{29}\right)^2} = \sqrt{\frac{116}{841} + \frac{725}{841}} = \sqrt{\frac{841}{841}} = 1.$$

CHECKPOINT Now try Exercise 37.

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle$$

as shown in Figure 6.26. (Note that the lowercase letter **i** is written in boldface to distinguish it from the imaginary number $i = \sqrt{-1}$.) These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2\rangle$ as follows.

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2\rangle \\ &= v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j}\end{aligned}$$

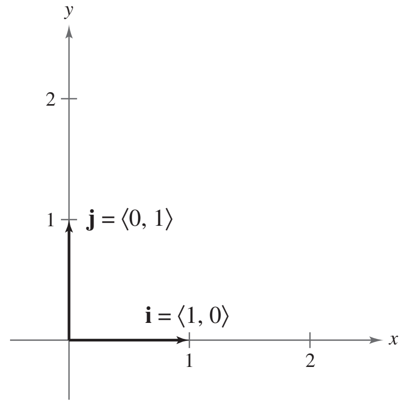


Figure 6.26

The scalars v_1 and v_2 are called the **horizontal and vertical components of v**, respectively. The vector sum

$$v_1\mathbf{i} + v_2\mathbf{j}$$

is called a **linear combination** of the vectors **i** and **j**. Any vector in the plane can be written as a linear combination of the standard unit vectors **i** and **j**.

Example 5 Writing a Linear Combination of Unit Vectors

Let **u** be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$. Write **u** as a linear combination of the standard unit vectors **i** and **j**.

Solution

Begin by writing the component form of the vector **u**.

$$\begin{aligned}\mathbf{u} &= \langle -1 - 2, 3 - (-5)\rangle \\ &= \langle -3, 8\rangle \\ &= -3\mathbf{i} + 8\mathbf{j}\end{aligned}$$

This result is shown graphically in Figure 6.27.

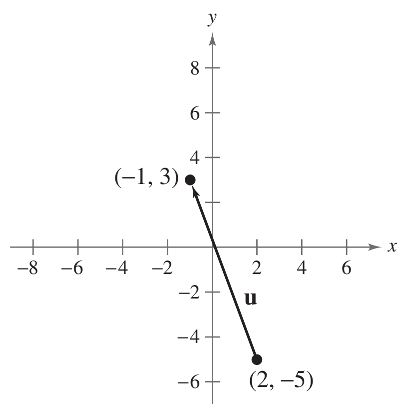


Figure 6.27

CHECKPOINT Now try Exercise 51.

Example 6 Vector Operations

Let $\mathbf{u} = -3\mathbf{i} + 8\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Find $2\mathbf{u} - 3\mathbf{v}$.

Solution

You could solve this problem by converting **u** and **v** to component form. This, however, is not necessary. It is just as easy to perform the operations in unit vector form.

$$\begin{aligned}2\mathbf{u} - 3\mathbf{v} &= 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) \\ &= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} \\ &= -12\mathbf{i} + 19\mathbf{j}\end{aligned}$$

CHECKPOINT Now try Exercise 57.

Direction Angles

If **u** is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive x-axis to **u**, the terminal point of **u** lies on the unit circle and you have

$$\mathbf{u} = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

as shown in Figure 6.28. The angle θ is the **direction angle** of the vector **u**.

Suppose that **u** is a unit vector with direction angle θ. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive x-axis, then it has the same direction as **u** and you can write

$$\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta, \sin\theta \rangle$$
$$= \|\mathbf{v}\| (\cos\theta)\mathbf{i} + \|\mathbf{v}\| (\sin\theta)\mathbf{j}.$$

Because $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\| (\cos\theta)\mathbf{i} + \|\mathbf{v}\| (\sin\theta)\mathbf{j}$, it follows that the direction angle θ for **v** is determined from

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{Quotient identity}$$

$$= \frac{\|\mathbf{v}\| \sin\theta}{\|\mathbf{v}\| \cos\theta} \qquad \text{Multiply numerator and denominator by } \|\mathbf{v}\|.$$

$$= \frac{b}{a}. \qquad \text{Simplify.}$$

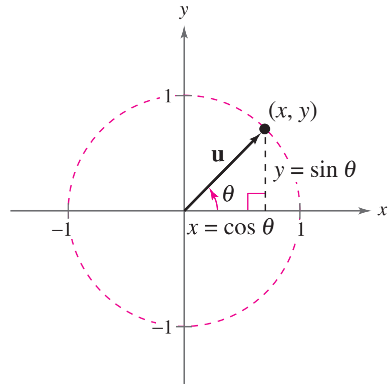


Figure 6.28

Example 7 Finding Direction Angles of Vectors

Find the direction angle of each vector.

a. $\mathbf{u} = 3\mathbf{i} + 3\mathbf{j}$ **b.** $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

Solution

a. The direction angle is

$$\tan\theta = \frac{b}{a} = \frac{3}{3} = 1.$$

So, $\theta = 45°$, as shown in Figure 6.29.

b. The direction angle is

$$\tan\theta = \frac{b}{a} = \frac{-4}{3}.$$

Moreover, because $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ lies in Quadrant IV, θ lies in Quadrant IV and its reference angle is

$$\theta' = \left|\arctan\left(-\frac{4}{3}\right)\right| \approx |-53.13°| = 53.13°.$$

So, it follows that $\theta \approx 360° - 53.13° = 306.87°$, as shown in Figure 6.30.

CHECKPOINT Now try Exercise 65.

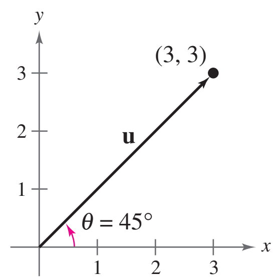


Figure 6.29

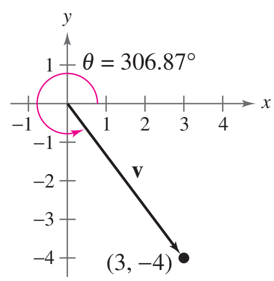


Figure 6.30

Applications of Vectors

Example 8 Finding the Component Form of a Vector

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle of 30° below the horizontal, as shown in Figure 6.31.

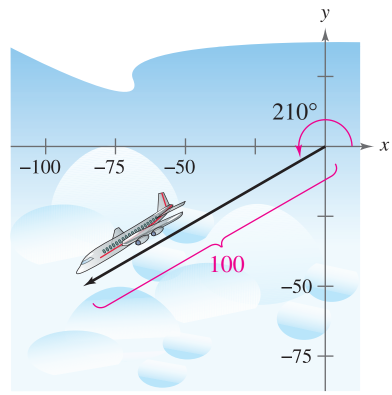


Figure 6.31

Solution

The velocity vector $\mathbf{v}$ has a magnitude of 100 and a direction angle of $\theta = 210°$.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}\\ &= 100(\cos 210°)\mathbf{i} + 100(\sin 210°)\mathbf{j}\\ &= 100\left(-\frac{\sqrt{3}}{2}\right)\mathbf{i} + 100\left(-\frac{1}{2}\right)\mathbf{j}\\ &= -50\sqrt{3}\,\mathbf{i} - 50\mathbf{j} = \langle -50\sqrt{3}, -50\rangle\end{aligned}$$

You can check that $\mathbf{v}$ has a magnitude of 100 as follows.

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-50\sqrt{3})^2 + (-50)^2}\\ &= \sqrt{7500 + 2500} = \sqrt{10{,}000} = 100\end{aligned}$$

Solution checks. ✓

CHECKPOINT Now try Exercise 83.

Example 9 Using Vectors to Determine Weight

A force of 600 pounds is required to pull a boat and trailer up a ramp inclined at 15° from the horizontal. Find the combined weight of the boat and trailer.

Solution

Based on Figure 6.32, you can make the following observations.

$\|\overrightarrow{BA}\|$ = force of gravity = combined weight of boat and trailer

$\|\overrightarrow{BC}\|$ = force against ramp

$\|\overrightarrow{AC}\|$ = force required to move boat up ramp = 600 pounds

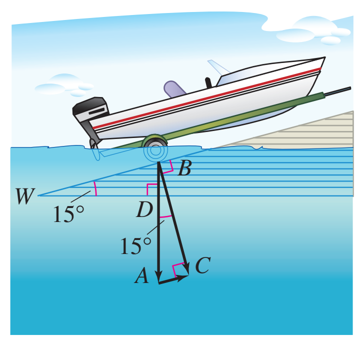


Figure 6.32

By construction, triangles BWD and ABC are similar. So, angle ABC is 15°. In triangle ABC you have

$$\sin 15° = \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{BA}\|} = \frac{600}{\|\overrightarrow{BA}\|}$$

$$\|\overrightarrow{BA}\| = \frac{600}{\sin 15°} \approx 2318.$$

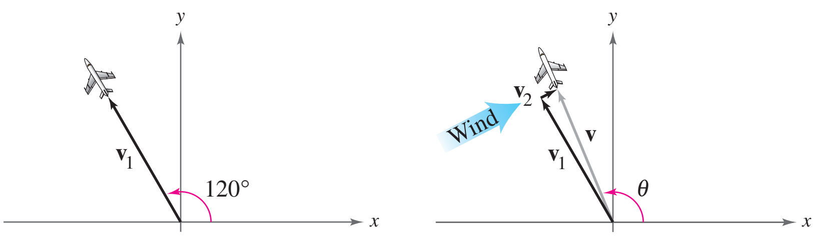
So, the combined weight is approximately 2318 pounds. (In Figure 6.32, note that $\overline{AC}$ is parallel to the ramp.)

CHECKPOINT Now try Exercise 85.

Example 10 Using Vectors to Find Speed and Direction

An airplane is traveling at a speed of 500 miles per hour with a bearing of 330° at a fixed altitude with a negligible wind velocity, as shown in Figure 6.33(a). As the airplane reaches a certain point, it encounters a wind blowing with a velocity of 70 miles per hour in the direction N 45° E, as shown in Figure 6.33(b). What are the resultant speed and direction of the airplane?

> **STUDY TIP**
>
> Recall from Section 4.8 that in air navigation, bearings are measured in degrees clockwise from north.



(a) (b)

Figure 6.33

Solution

Using Figure 6.33, the velocity of the airplane (alone) is

$$\begin{aligned}\mathbf{v}_1 &= 500\langle\cos 120°, \sin 120°\rangle\\ &= \langle -250, 250\sqrt{3}\rangle\end{aligned}$$

and the velocity of the wind is

$$\begin{aligned}\mathbf{v}_2 &= 70\langle\cos 45°, \sin 45°\rangle\\ &= \langle 35\sqrt{2}, 35\sqrt{2}\rangle.\end{aligned}$$

So, the velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2\\ &= \langle -250 + 35\sqrt{2}, 250\sqrt{3} + 35\sqrt{2}\rangle\\ &\approx \langle -200.5, 482.5\rangle\end{aligned}$$

and the resultant speed of the airplane is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-200.5)^2 + (482.5)^2}\\ &\approx 522.5 \text{ miles per hour.}\end{aligned}$$

Finally, if θ is the direction angle of the flight path, you have

$$\tan\theta = \frac{482.5}{-200.5} \approx -2.4065$$

which implies that

$$\theta \approx 180° + \arctan(-2.4065) \approx 180° - 67.4° = 112.6°.$$

So, the true direction of the airplane is 337.4°.

CHECKPOINT Now try Exercise 89.

6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

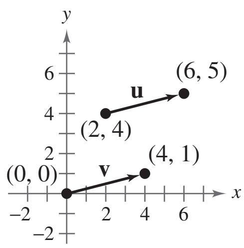
Vocabulary Check

Fill in the blanks.

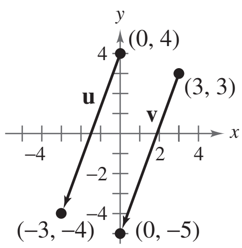
1. A _______ can be used to represent a quantity that involves both magnitude and direction.
2. The directed line segment $\overrightarrow{PQ}$ has _______ point P and _______ point Q.
3. The _______ of the directed line segment $\overrightarrow{PQ}$ is denoted by $\|\overrightarrow{PQ}\|$.
4. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a _______ $\mathbf{v}$ in the plane.
5. The directed line segment whose initial point is the origin is said to be in _______ .
6. A vector that has a magnitude of 1 is called a _______ .
7. The two basic vector operations are scalar _______ and vector _______ .
8. The vector $\mathbf{u} + \mathbf{v}$ is called the _______ of vector addition.
9. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is called a _______ of the vectors $\mathbf{i}$ and $\mathbf{j}$, and the scalars v_1 and v_2 are called the _______ and _______ components of $\mathbf{v}$, respectively.

In Exercises 1 and 2, show that $\mathbf{u} = \mathbf{v}$.

1.

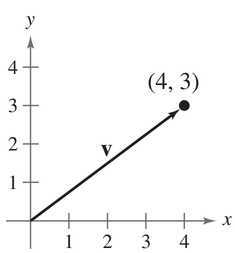


2.

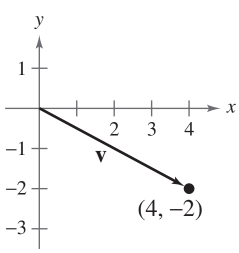


In Exercises 3–12, find the component form and the magnitude of the vector v.

3.



4.



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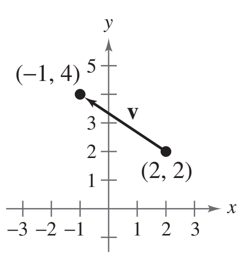

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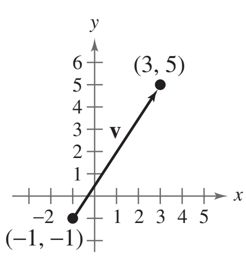

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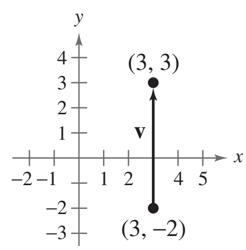
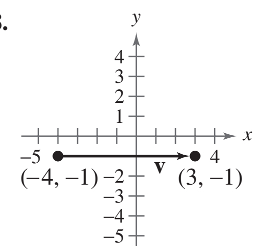
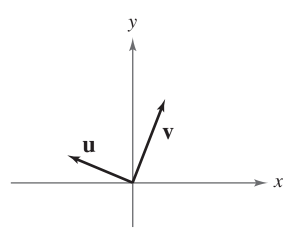


8.



	Initial Point	*Terminal Point*
9.	$\left(\frac{2}{5}, 1\right)$	$\left(1, \frac{2}{5}\right)$
10.	$\left(\frac{7}{2}, 0\right)$	$\left(0, -\frac{7}{2}\right)$
11.	$\left(-\frac{2}{3}, -1\right)$	$\left(\frac{1}{2}, \frac{4}{5}\right)$
12.	$\left(\frac{5}{2}, -2\right)$	$\left(1, \frac{2}{5}\right)$

In Exercises 13–18, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



13. $-\mathbf{v}$

14. $3\mathbf{u}$

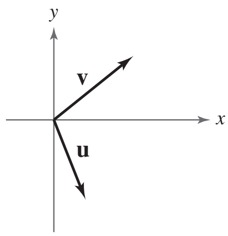
15. $\mathbf{u} + \mathbf{v}$

16. $\mathbf{u} - \mathbf{v}$

17. $\mathbf{u} + 2\mathbf{v}$

18. $\mathbf{v} - \frac{1}{2}\mathbf{u}$

In Exercises 19–24, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

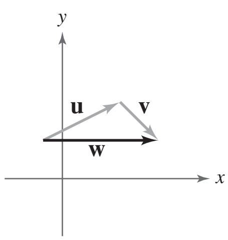


19. $2\mathbf{u}$ **20.** $-3\mathbf{v}$
21. $\mathbf{u} + 2\mathbf{v}$ **22.** $\frac{1}{2}\mathbf{v}$
23. $\mathbf{v} - \frac{1}{2}\mathbf{u}$ **24.** $2\mathbf{u} + 3\mathbf{v}$

In Exercises 25–30, find (a) u + v, (b) u − v, (c) 2u − 3v, and (d) v + 4u.

25. $\mathbf{u} = \langle 4, 2 \rangle,\ \mathbf{v} = \langle 7, 1 \rangle$ **26.** $\mathbf{u} = \langle 5, 3 \rangle,\ \mathbf{v} = \langle -4, 0 \rangle$
27. $\mathbf{u} = \langle -6, -8 \rangle,\ \mathbf{v} = \langle 2, 4 \rangle$
28. $\mathbf{u} = \langle 0, -5 \rangle,\ \mathbf{v} = \langle -3, 9 \rangle$
29. $\mathbf{u} = \mathbf{i} + \mathbf{j},\ \mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ **30.** $\mathbf{u} = 2\mathbf{i} - \mathbf{j},\ \mathbf{v} = -\mathbf{i} + \mathbf{j}$

In Exercises 31–34, use the figure and write the vector in terms of the other two vectors.



31. $\mathbf{w}$ **32.** $\mathbf{v}$
33. $\mathbf{u}$ **34.** $2\mathbf{v}$

In Exercises 35–44, find a unit vector in the direction of the given vector.

35. $\mathbf{u} = \langle 6, 0 \rangle$ **36.** $\mathbf{u} = \langle 0, -2 \rangle$
37. $\mathbf{v} = \langle -1, 1 \rangle$ **38.** $\mathbf{v} = \langle 3, -4 \rangle$
39. $\mathbf{v} = \langle -24, -7 \rangle$ **40.** $\mathbf{v} = \langle 8, -20 \rangle$
41. $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$ **42.** $\mathbf{w} = \mathbf{i} - 2\mathbf{j}$
43. $\mathbf{w} = 2\mathbf{j}$ **44.** $\mathbf{w} = -3\mathbf{i}$

In Exercises 45–50, find the vector v with the given magnitude and the same direction as u.

	Magnitude	*Direction*
45.	$\|\mathbf{v}\| = 8$	$\mathbf{u} = \langle 5, 6 \rangle$
46.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 4, -4 \rangle$
47.	$\|\mathbf{v}\| = 7$	$\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$
48.	$\|\mathbf{v}\| = 10$	$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$
49.	$\|\mathbf{v}\| = 8$	$\mathbf{u} = -2\mathbf{i}$
50.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = 5\mathbf{j}$

In Exercises 51–54, the initial and terminal points of a vector are given. Write the vector as a linear combination of the standard unit vectors i and j.

	Initial Point	*Terminal Point*
51.	$(-3, 1)$	$(4, 5)$
52.	$(0, -2)$	$(3, 6)$
53.	$(-1, -5)$	$(2, 3)$
54.	$(-6, 4)$	$(0, 1)$

In Exercises 55–60, find the component form of v and sketch the specified vector operations geometrically, where u = 2i − j and w = i + 2j.

55. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **56.** $\mathbf{v} = \frac{2}{3}\mathbf{w}$
57. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ **58.** $\mathbf{v} = -\mathbf{u} + \mathbf{w}$
59. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$ **60.** $\mathbf{v} = 2\mathbf{u} - 2\mathbf{w}$

In Exercises 61–66, find the magnitude and direction angle of the vector v.

61. $\mathbf{v} = 5(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$
62. $\mathbf{v} = 8(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j})$
63. $\mathbf{v} = 6\mathbf{i} - 6\mathbf{j}$ **64.** $\mathbf{v} = -4\mathbf{i} - 7\mathbf{j}$
65. $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$ **66.** $\mathbf{v} = 12\mathbf{i} + 15\mathbf{j}$

In Exercises 67–72, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis. Sketch v.

	Magnitude	*Angle*
67.	$\|\mathbf{v}\| = 3$	$\theta = 0^\circ$
68.	$\|\mathbf{v}\| = 1$	$\theta = 45^\circ$
69.	$\|\mathbf{v}\| = 3\sqrt{2}$	$\theta = 150^\circ$
70.	$\|\mathbf{v}\| = 4\sqrt{3}$	$\theta = 90^\circ$
71.	$\|\mathbf{v}\| = 2$	$\mathbf{v}$ in the direction $\mathbf{i} + 3\mathbf{j}$
72.	$\|\mathbf{v}\| = 3$	$\mathbf{v}$ in the direction $3\mathbf{i} + 4\mathbf{j}$

In Exercises 73–76, find the component form of the sum of u and v with direction angles $\theta_{\mathbf{u}}$ and $\theta_{\mathbf{v}}$.

	Magnitude	*Angle*
73.	$\|\mathbf{u}\| = 5$	$\theta_{\mathbf{u}} = 60^\circ$
	$\|\mathbf{v}\| = 5$	$\theta_{\mathbf{v}} = 90^\circ$

	Magnitude	*Angle*
74.	$\|\mathbf{u}\| = 2$	$\theta_{\mathbf{u}} = 30°$
	$\|\mathbf{v}\| = 2$	$\theta_{\mathbf{v}} = 90°$
75.	$\|\mathbf{u}\| = 20$	$\theta_{\mathbf{u}} = 45°$
	$\|\mathbf{v}\| = 50$	$\theta_{\mathbf{v}} = 150°$
76.	$\|\mathbf{u}\| = 35$	$\theta_{\mathbf{u}} = 25°$
	$\|\mathbf{v}\| = 50$	$\theta_{\mathbf{v}} = 120°$

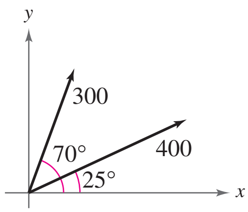
In Exercises 77 and 78, use the Law of Cosines to find the angle α between the vectors. (Assume $0° \le \alpha \le 180°$.)

77. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = 2(\mathbf{i} - \mathbf{j})$

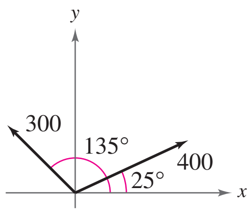
78. $\mathbf{v} = 3\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = 2\mathbf{i} - \mathbf{j}$

In Exercises 79 and 80, graph the vectors and the resultant of the vectors. Find the magnitude and direction of the resultant.

79.



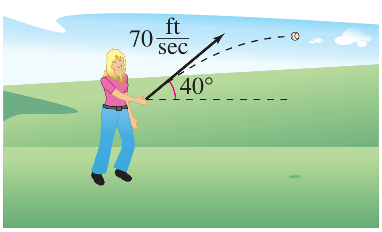
80.



Resultant Force **In Exercises 81 and 82, find the angle between the forces given the magnitude of their resultant. (*Hint:* Write force 1 as a vector in the direction of the positive x-axis and force 2 as a vector at an angle θ with the positive x-axis.)**

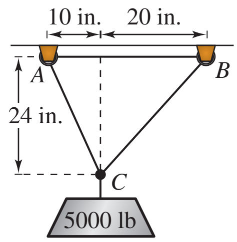
	Force 1	*Force 2*	*Resultant Force*
81.	45 pounds	60 pounds	90 pounds
82.	3000 pounds	1000 pounds	3750 pounds

83. ***Velocity*** A ball is thrown with an initial velocity of 70 feet per second, at an angle of 40° with the horizontal (see figure). Find the vertical and horizontal components of the velocity.

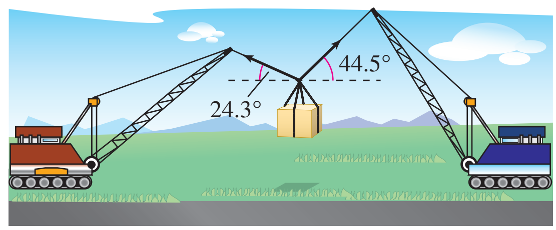


84. ***Velocity*** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 4° with the horizontal. Find the vertical and horizontal components of the velocity.

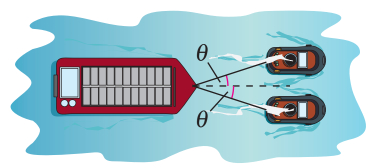
85. ***Tension*** Use the figure to determine the tension in each cable supporting the load.



86. ***Tension*** The cranes shown in the figure are lifting an object that weighs 20,240 pounds. Find the tension in the cable of each crane.



87. ***Numerical and Graphical Analysis*** A loaded barge is being towed by two tugboats, and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Each tow line makes an angle of θ degrees with the axis of the barge.



(a) Write the resultant tension T of each tow line as a function of θ. Determine the domain of the function.

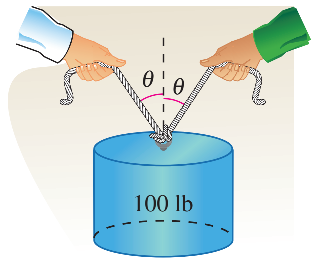
(b) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°	60°
T						

(c) Use a graphing utility to graph the tension function.

(d) Explain why the tension increases as θ increases.

88. ***Numerical and Graphical Analysis*** To carry a 100-pound cylindrical weight, two people lift on the ends of short ropes that are tied to an eyelet on the top center of the cylinder. Each rope makes an angle of θ degrees with the vertical (see figure).



(a) Write the tension T of each rope as a function of θ. Determine the domain of the function.

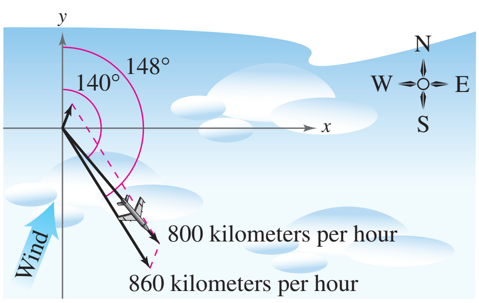
(b) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°	60°
T						

(c) Use a graphing utility to graph the tension function.

(d) Explain why the tension increases as θ increases.

89. ***Navigation*** An airplane is flying in the direction 148° with an airspeed of 860 kilometers per hour. Because of the wind, its groundspeed and direction are, respectively, 800 kilometers per hour and 140°. Find the direction and speed of the wind.



90. ***Navigation*** A commercial jet is flying from Miami to Seattle. The jet's velocity with respect to the air is 580 miles per hour, and its bearing is 332°. The wind, at the altitude of the plane, is blowing from the southwest with a velocity of 60 miles per hour.

(a) Draw a figure that gives a visual representation of the problem.

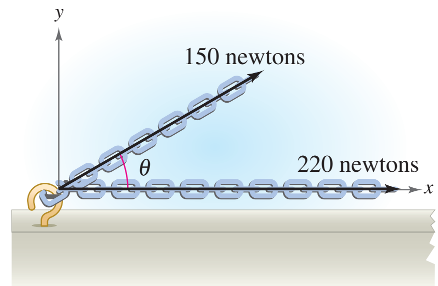
(b) Write the velocity of the wind as a vector in component form.

(c) Write the velocity of the jet relative to the air as a vector in component form.

(d) What is the speed of the jet with respect to the ground?

(e) What is the true direction of the jet?

91. ***Numerical and Graphical Analysis*** Forces with magnitudes of 150 newtons and 220 newtons act on a hook (see figure).



(a) Find the direction and magnitude of the resultant of the forces when $\theta = 30°$.

(b) Write the magnitude M of the resultant and the direction α of the resultant as functions of θ, where $0° \le \theta \le 180°$.

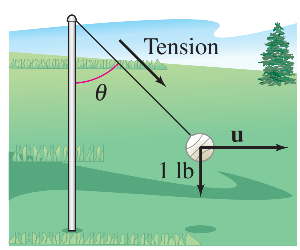
(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions.

(e) Explain why one function decreases for increasing θ, whereas the other doesn't.

92. ***Numerical and Graphical Analysis*** A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force **u** until the rope makes an angle of θ degrees with the pole (see figure).



(a) Write the tension T in the rope and the magnitude of **u** as functions of θ. Determine the domains of the functions.

(b) Use a graphing utility to complete the table.

θ	$0°$	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$
T							
$\lVert\mathbf{u}\rVert$							

(c) Use a graphing utility to graph the two functions for $0° \le \theta \le 60°$.

(d) Compare T and $\lVert\mathbf{u}\rVert$ as θ increases.

Synthesis

True or False? **In Exercises 93–96, determine whether the statement is true or false. Justify your answer.**

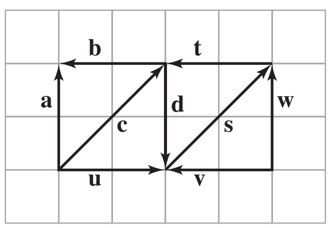
93. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction, then $\mathbf{u} = \mathbf{v}$.

94. If $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$, then $\mathbf{v} = \lVert\mathbf{v}\rVert\,\mathbf{u}$.

95. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

96. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

True or False? **In Exercises 97–104, use the figure to determine whether the statement is true or false. Justify your answer.**



97. $\mathbf{a} = -\mathbf{d}$

98. $\mathbf{c} = \mathbf{s}$

99. $\mathbf{a} + \mathbf{u} = \mathbf{c}$

100. $\mathbf{v} + \mathbf{w} = -\mathbf{s}$

101. $\mathbf{a} + \mathbf{w} = -2\mathbf{d}$

102. $\mathbf{a} + \mathbf{d} = \mathbf{0}$

103. $\mathbf{u} - \mathbf{v} = -2(\mathbf{b} + \mathbf{t})$

104. $\mathbf{t} - \mathbf{w} = \mathbf{b} - \mathbf{a}$

105. ***Think About It*** Consider two forces of equal magnitude acting on a point.

(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

(b) If the resultant of the forces is $\mathbf{0}$, make a conjecture about the angle between the forces.

(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

106. ***Graphical Reasoning*** Consider two forces

$$\mathbf{F}_1 = \langle 10, 0\rangle \quad \text{and} \quad \mathbf{F}_2 = 5\langle \cos\theta, \sin\theta\rangle.$$

(a) Find $\lVert\mathbf{F}_1 + \mathbf{F}_2\rVert$ as a function of θ.

(b) Use a graphing utility to graph the function for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum, and for what value of θ does it occur? What is its minimum, and for what value of θ does it occur?

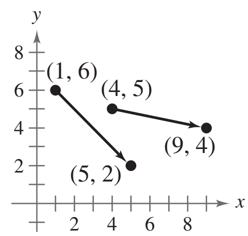
(d) Explain why the magnitude of the resultant is never 0.

107. ***Proof*** Prove that $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ is a unit vector for any value of θ.

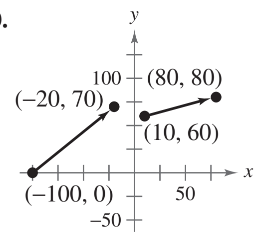
108. ***Technology*** Write a program for your graphing utility that graphs two vectors and their difference given the vectors in component form.

In Exercises 109 and 110, use the program in Exercise 108 to find the difference of the vectors shown in the graph.

109.



110.



Skills Review

In Exercises 111–116, simplify the expression.

111. $\left(\dfrac{6x^4}{7y^{-2}}\right)(14x^{-1}y^5)$

112. $(5s^5t^{-5})\left(\dfrac{3s^{-2}}{50t^{-1}}\right)$

113. $(18x)^0(4xy)^2(3x^{-1})$

114. $(5ab^2)(a^{-3}b^0)(2a^0b)^{-2}$

115. $(2.1 \times 10^9)(3.4 \times 10^{-4})$

116. $(6.5 \times 10^6)(3.8 \times 10^4)$

In Exercises 117–120, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

117. $\sqrt{49 - x^2}, \quad x = 7\sin\theta$

118. $\sqrt{x^2 - 49}, \quad x = 7\sec\theta$

119. $\sqrt{x^2 + 100}, \quad x = 10\cot\theta$

120. $\sqrt{x^2 - 4}, \quad x = 2\csc\theta$

In Exercises 121–124, solve the equation.

121. $\cos x(\cos x + 1) = 0$

122. $\sin x\left(2\sin x + \sqrt{2}\right) = 0$

123. $3\sec x + 4 = 10$

124. $\cos x \cot x - \cos x = 0$

6.4 Vectors and Dot Products

What you should learn

- Find the dot product of two vectors and use properties of the dot product.
- Find angles between vectors and determine whether two vectors are orthogonal.
- Write vectors as sums of two vector components.
- Use vectors to find the work done by a force.

Why you should learn it

You can use the dot product of two vectors to solve real-life problems involving two vector quantities. For instance, Exercise 61 on page 446 shows you how the dot product can be used to find the force necessary to keep a truck from rolling down a hill.

Alan Thornton/Getty Images

The Dot Product of Two Vectors

So far you have studied two vector operations—vector addition and multiplication by a scalar—each of which yields another vector. In this section, you will study a third vector operation, the **dot product.** This product yields a scalar, rather than a vector.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Properties of the Dot Product (See the proofs on page 471.)

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Example 1 Finding Dot Products

Find each dot product.

a. $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle$

b. $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle$

c. $\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle$

Solution

a. $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle = 4(2) + 5(3) = 8 + 15 = 23$

b. $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle = 2(1) + (-1)(2) = 2 - 2 = 0$

c. $\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle = 0(4) + 3(-2) = 0 - 6 = -6$

CHECKPOINT Now try Exercise 1.

In Example 1, be sure you see that the dot product of two vectors is a scalar (a real number), not a vector. Moreover, notice that the dot product can be positive, zero, or negative.

Example 2 Using Properties of Dot Products

Let $\mathbf{u} = \langle -1, 3\rangle$, $\mathbf{v} = \langle 2, -4\rangle$, and $\mathbf{w} = \langle 1, -2\rangle$. Find each dot product.

a. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ **b.** $\mathbf{u} \cdot 2\mathbf{v}$

Solution

Begin by finding the dot product of $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle -1, 3\rangle \cdot \langle 2, -4\rangle \\ &= (-1)(2) + 3(-4) \\ &= -14\end{aligned}$$

a. $$\begin{aligned}(\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= -14\langle 1, -2\rangle \\ &= \langle -14, 28\rangle\end{aligned}$$

b. $$\begin{aligned}\mathbf{u} \cdot 2\mathbf{v} &= 2(\mathbf{u} \cdot \mathbf{v}) \\ &= 2(-14) \\ &= -28\end{aligned}$$

Notice that the product in part (a) is a vector, whereas the product in part (b) is a scalar. Can you see why?

CHECKPOINT Now try Exercise 9.

Example 3 Dot Product and Magnitude

The dot product of $\mathbf{u}$ with itself is 5. What is the magnitude of $\mathbf{u}$?

Solution

Because $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 5$, it follows that

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{\mathbf{u} \cdot \mathbf{u}} \\ &= \sqrt{5}.\end{aligned}$$

CHECKPOINT Now try Exercise 11.

The Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 6.34. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

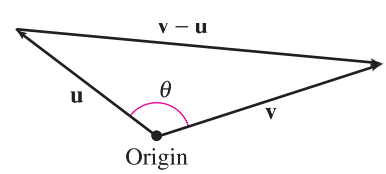


Figure 6.34

Angle Between Two Vectors (See the proof on page 471.)

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Example 4 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 3, 5 \rangle$.

Solution

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$= \frac{\langle 4, 3\rangle \cdot \langle 3, 5\rangle}{\|\langle 4, 3\rangle\|\,\|\langle 3, 5\rangle\|}$$

$$= \frac{27}{5\sqrt{34}}$$

This implies that the angle between the two vectors is

$$\theta = \arccos\frac{27}{5\sqrt{34}} \approx 22.2^\circ$$

as shown in Figure 6.35.

CHECKPOINT Now try Exercise 17.

TECHNOLOGY TIP

The graphing utility program Finding the Angle Between Two Vectors, found at this textbook's *Online Study Center*, graphs two vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$ in standard position and finds the measure of the angle between them. Use the program to verify Example 4.

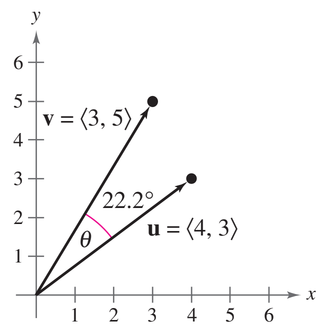
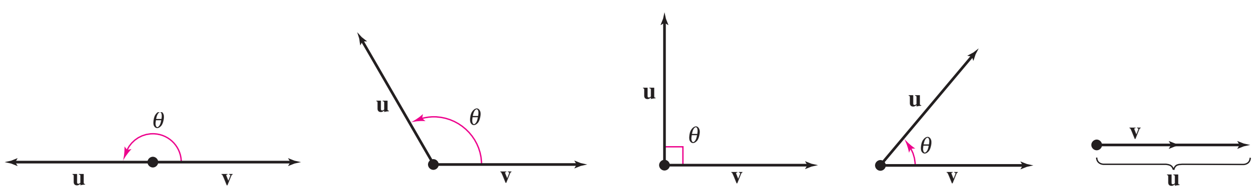


Figure 6.35

Rewriting the expression for the angle between two vectors in the form

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \qquad \text{Alternative form of dot product}$$

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u}\cdot\mathbf{v}$ and $\cos\theta$ will always have the same sign. Figure 6.36 shows the five possible orientations of two vectors.



$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos\theta = -1$	$-1 < \cos\theta < 0$	$\cos\theta = 0$	$0 < \cos\theta < 1$	$\cos\theta = 1$
Opposite direction	*Obtuse angle*	*90° angle*	*Acute angle*	*Same direction*

Figure 6.36

Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$.

The terms *orthogonal* and *perpendicular* mean essentially the same thing—meeting at right angles. Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the zero vector is orthogonal to every vector $\mathbf{u}$ because $\mathbf{0}\cdot\mathbf{u} = 0$.

Example 5 Determining Orthogonal Vectors

Are the vectors $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle 6, 4 \rangle$ orthogonal?

Solution

Begin by finding the dot product of the two vectors.

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, -3 \rangle \cdot \langle 6, 4 \rangle = 2(6) + (-3)(4) = 0$$

Because the dot product is 0, the two vectors are orthogonal, as shown in Figure 6.37.

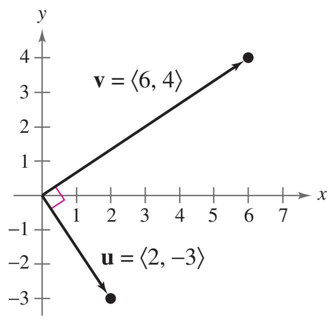


Figure 6.37

CHECKPOINT Now try Exercise 35.

Finding Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components.**

Consider a boat on an inclined ramp, as shown in Figure 6.38. The force $\mathbf{F}$ due to gravity pulls the boat *down* the ramp and *against* the ramp. These two orthogonal forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are vector components of $\mathbf{F}$. That is,

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2. \qquad \text{Vector components of } \mathbf{F}$$

The negative of component $\mathbf{w}_1$ represents the force needed to keep the boat from rolling down the ramp, and $\mathbf{w}_2$ represents the force that the tires must withstand against the ramp. A procedure for finding $\mathbf{w}_1$ and $\mathbf{w}_2$ is shown on the next page.

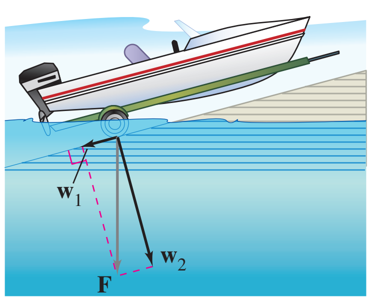


Figure 6.38

Definition of Vector Components

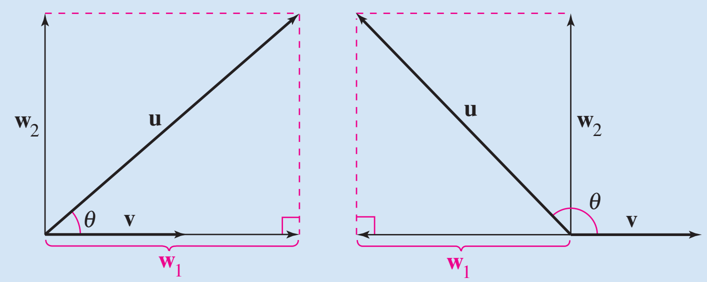
Let **u** and **v** be nonzero vectors such that

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to (or a scalar multiple of) **v**, as shown in Figure 6.39. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are called **vector components** of **u**. The vector $\mathbf{w}_1$ is the **projection** of **u** onto **v** and is denoted by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}.$$

The vector $\mathbf{w}_2$ is given by $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.



θ is acute. *θ is obtuse*

Figure 6.39

From the definition of vector components, you can see that it is easy to find the component $\mathbf{w}_2$ once you have found the projection of **u** onto **v**. To find the projection, you can use the dot product, as follows.

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2$$ $\mathbf{w}_1$ is a scalar multiple of **v**.

$$\mathbf{u} \cdot \mathbf{v} = (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v}$$ Take dot product of each side with **v**.

$$= c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v}$$

$$= c\|\mathbf{v}\|^2 + 0$$ $\mathbf{w}_2$ and **v** are orthogonal.

So,

$$c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

and

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$

Projection of u onto v

Let **u** and **v** be nonzero vectors. The projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Example 6 Decomposing a Vector into Components

Find the projection of $\mathbf{u} = \langle 3, -5 \rangle$ onto $\mathbf{v} = \langle 6, 2 \rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution

The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{8}{40}\right)\langle 6, 2 \rangle = \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle$$

as shown in Figure 6.40. The other component, $\mathbf{w}_2$, is

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 3, -5 \rangle - \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle = \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle.$$

So, $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle + \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle = \langle 3, -5 \rangle.$

CHECKPOINT Now try Exercise 45.

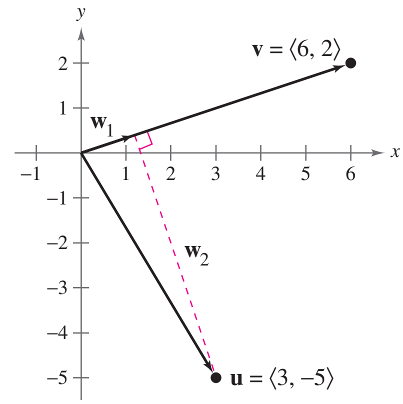


Figure 6.40

Example 7 Finding a Force

A 200-pound cart sits on a ramp inclined at 30°, as shown in Figure 6.41. What force is required to keep the cart from rolling down the ramp?

Solution

Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector

$$\mathbf{F} = -200\mathbf{j}. \qquad \text{Force due to gravity}$$

To find the force required to keep the cart from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = (\cos 30°)\mathbf{i} + (\sin 30°)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \qquad \text{Unit vector along ramp}$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is

$$\begin{aligned}\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\ &= (\mathbf{F} \cdot \mathbf{v})\mathbf{v} \\ &= (-200)\left(\frac{1}{2}\right)\mathbf{v} \\ &= -100\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).\end{aligned}$$

The magnitude of this force is 100, and therefore a force of 100 pounds is required to keep the cart from rolling down the ramp.

CHECKPOINT Now try Exercise 61.

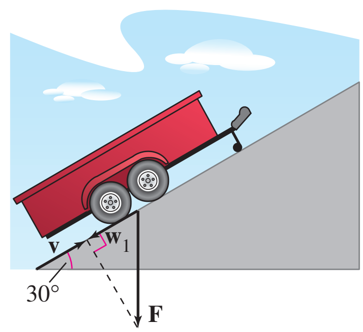


Figure 6.41

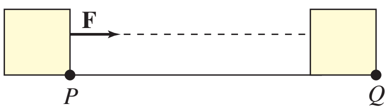
Work

The work W done by a constant force $\mathbf{F}$ acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\| \, \|\overrightarrow{PQ}\|$$

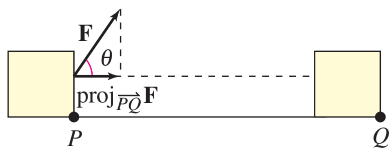
as shown in Figure 6.42. If the constant force $\mathbf{F}$ is not directed along the line of motion (see Figure 6.43), the work W done by the force is given by

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \, \|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= (\cos\theta) \|\mathbf{F}\| \, \|\overrightarrow{PQ}\| && \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| = (\cos\theta)\|\mathbf{F}\| \\ &= \mathbf{F} \cdot \overrightarrow{PQ}. && \text{Dot product form for work} \end{aligned}$$



Force acts along the line of motion.

Figure 6.42



Force acts at angle θ with the line of motion.

Figure 6.43

This notion of work is summarized in the following definition.

Definition of Work

The **work** W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \, \|\overrightarrow{PQ}\|$ — Projection form

2. $W = \mathbf{F} \cdot \overrightarrow{PQ}$ — Dot product form

Example 8 Finding Work

To close a barn's sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 6.44. Find the work done in moving the door 12 feet to its closed position.

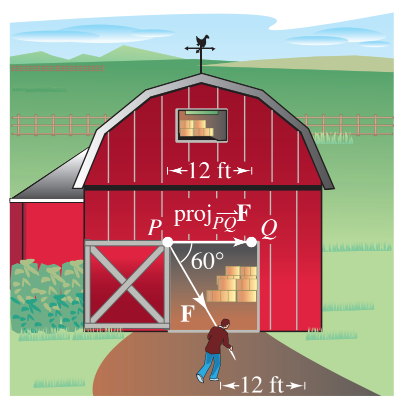


Figure 6.44

Solution

Using a projection, you can calculate the work as follows.

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}} \mathbf{F}\| \|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= (\cos 60°)\|\mathbf{F}\| \, \|\overrightarrow{PQ}\| \\ &= \frac{1}{2}(50)(12) = 300 \text{ foot-pounds} \end{aligned}$$

So, the work done is 300 foot-pounds. You can verify this result by finding the vectors $\mathbf{F}$ and $\overrightarrow{PQ}$ and calculating their dot product.

✓CHECKPOINT Now try Exercise 63.

6.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The _______ of two vectors yields a scalar, rather than a vector.

2. If θ is the angle between two nonzero vectors **u** and **v**, then $\cos\theta =$ _______ .

3. The vectors **u** and **v** are _______ if $\mathbf{u}\cdot\mathbf{v} = 0$.

4. The projection of **u** onto **v** is given by $\text{proj}_{\mathbf{v}}\mathbf{u} =$ _______ .

5. The work W done by a constant force **F** as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either $W =$ ______ or $W =$ _______ .

In Exercises 1–4, find the dot product of u and v.

1. $\mathbf{u} = \langle 6, 3\rangle$, $\mathbf{v} = \langle 2, -4\rangle$

2. $\mathbf{u} = \langle -4, 1\rangle$, $\mathbf{v} = \langle 2, -3\rangle$

3. $\mathbf{u} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$

4. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$

In Exercises 5–10, use the vectors $\mathbf{u} = \langle 2, 2\rangle$, $\mathbf{v} = \langle -3, 4\rangle$, and $\mathbf{w} = \langle 1, -4\rangle$ to find the indicated quantity. State whether the result is a vector or a scalar.

5. $\mathbf{u}\cdot\mathbf{u}$

6. $\mathbf{v}\cdot\mathbf{w}$

7. $\mathbf{u}\cdot 2\mathbf{v}$

8. $4\mathbf{u}\cdot\mathbf{v}$

9. $(3\mathbf{w}\cdot\mathbf{v})\mathbf{u}$

10. $(\mathbf{u}\cdot 2\mathbf{v})\mathbf{w}$

In Exercises 11–16, use the dot product to find the magnitude of u.

11. $\mathbf{u} = \langle -5, 12\rangle$

12. $\mathbf{u} = \langle 2, -4\rangle$

13. $\mathbf{u} = 20\mathbf{i} + 25\mathbf{j}$

14. $\mathbf{u} = 6\mathbf{i} - 10\mathbf{j}$

15. $\mathbf{u} = -4\mathbf{j}$

16. $\mathbf{u} = 9\mathbf{i}$

In Exercises 17–24, find the angle θ between the vectors.

17. $\mathbf{u} = \langle -1, 0\rangle$, $\mathbf{v} = \langle 0, 2\rangle$

18. $\mathbf{u} = \langle 4, 4\rangle$, $\mathbf{v} = \langle -2, 0\rangle$

19. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$

20. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

21. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$

22. $\mathbf{u} = 4\mathbf{j}$, $\mathbf{v} = -3\mathbf{i}$

23. $\mathbf{u} = \cos\left(\frac{\pi}{3}\right)\mathbf{i} + \sin\left(\frac{\pi}{3}\right)\mathbf{j}$

$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$

24. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j}$

$\mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j}$

In Exercises 25–28, graph the vectors and find the degree measure of the angle between the vectors.

25. $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$

26. $\mathbf{u} = -6\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = -8\mathbf{i} + 4\mathbf{j}$

27. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 8\mathbf{i} - 5\mathbf{j}$

28. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$

In Exercises 29 and 30, use vectors to find the interior angles of the triangle with the given vertices.

29. $(1, 2), (3, 4), (2, 5)$

30. $(-3, 0), (2, 2), (0, 6)$

In Exercises 31 and 32, find $\mathbf{u}\cdot\mathbf{v}$, where θ is the angle between u and v.

31. $\|\mathbf{u}\| = 9, \|\mathbf{v}\| = 36, \theta = \frac{3\pi}{4}$

32. $\|\mathbf{u}\| = 4, \|\mathbf{v}\| = 12, \theta = \frac{\pi}{3}$

In Exercises 33–38, determine whether u and v are orthogonal, parallel, or neither.

33. $\mathbf{u} = \langle -12, 30\rangle$, $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{5}{4}\right\rangle$

34. $\mathbf{u} = \langle 15, 45\rangle$, $\mathbf{v} = \langle -5, 12\rangle$

35. $\mathbf{u} = \frac{1}{4}(3\mathbf{i} - \mathbf{j})$, $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

36. $\mathbf{u} = \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

37. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}$
$\mathbf{v} = -\mathbf{i} - \mathbf{j}$

38. $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j}$
$\mathbf{v} = -2\mathbf{i} - \mathbf{j}$

In Exercises 39–44, find the value of k so that the vectors u and v are orthogonal.

39. $\mathbf{u} = 2\mathbf{i} - k\mathbf{j}$
$\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$

40. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$
$\mathbf{v} = 2\mathbf{i} - k\mathbf{j}$

41. $\mathbf{u} = \mathbf{i} + 4\mathbf{j}$
$\mathbf{v} = 2k\mathbf{i} - 5\mathbf{j}$

42. $\mathbf{u} = -3k\mathbf{i} + 5\mathbf{j}$
$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

43. $\mathbf{u} = -3k\mathbf{i} + 2\mathbf{j}$
$\mathbf{v} = -6\mathbf{i}$

44. $\mathbf{u} = 4\mathbf{i} - 4k\mathbf{j}$
$\mathbf{v} = 3\mathbf{j}$

In Exercises 45–48, find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

45. $\mathbf{u} = \langle 3, 4\rangle$
$\mathbf{v} = \langle 8, 2\rangle$

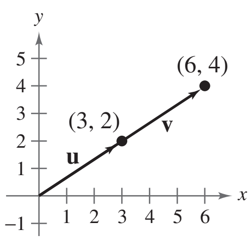
46. $\mathbf{u} = \langle 4, 2\rangle$
$\mathbf{v} = \langle 1, -2\rangle$

47. $\mathbf{u} = \langle 0, 3\rangle$
$\mathbf{v} = \langle 2, 15\rangle$

48. $\mathbf{u} = \langle -5, -1\rangle$
$\mathbf{v} = \langle -1, 1\rangle$

In Exercises 49–52, use the graph to determine mentally the projection of u onto v. (The coordinates of the terminal points of the vectors in standard position are given.) Use the formula for the projection of u onto v to verify your result.

49.



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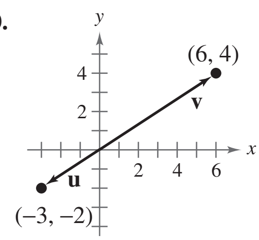

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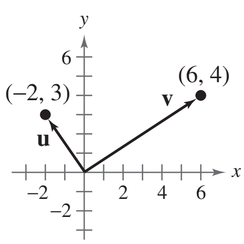
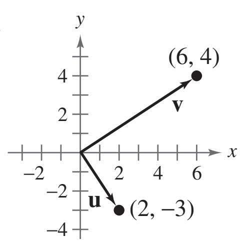


52.



In Exercises 53–56, find two vectors in opposite directions that are orthogonal to the vector u. (There are many correct answers.)

53. $\mathbf{u} = \langle 2, 6\rangle$

54. $\mathbf{u} = \langle -7, 5\rangle$

55. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j}$

56. $\mathbf{u} = -\frac{5}{2}\mathbf{i} - 3\mathbf{j}$

Work **In Exercises 57 and 58, find the work done in moving a particle from P to Q if the magnitude and direction of the force are given by v.**

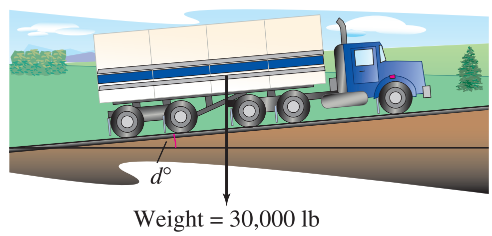
57. $P = (0, 0),\quad Q = (4, 7),\quad \mathbf{v} = \langle 1, 4\rangle$

58. $P = (1, 3),\quad Q = (-3, 5),\quad \mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$

59. ***Revenue*** The vector $\mathbf{u} = \langle 1245, 2600\rangle$ gives the numbers of units of two types of picture frames produced by a company. The vector $\mathbf{v} = \langle 12.20, 8.50\rangle$ gives the price (in dollars) of each frame, respectively. (a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and explain its meaning in the context of the problem. (b) Identify the vector operation used to increase prices by 2 percent.

60. ***Revenue*** The vector $\mathbf{u} = \langle 3240, 2450\rangle$ gives the numbers of hamburgers and hot dogs, respectively, sold at a fast food stand in one week. The vector $\mathbf{v} = \langle 1.75, 1.25\rangle$ gives the prices in dollars for the food items. (a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and explain its meaning in the context of the problem. (b) Identify the vector operation used to increase prices by $2\frac{1}{2}$ percent.

61. ***Braking Load*** A truck with a gross weight of 30,000 pounds is parked on a slope of $d°$ (see figure). Assume that the only force to overcome is the force of gravity.



(a) Find the force required to keep the truck from rolling down the hill in terms of the slope d.

(b) Use a graphing utility to complete the table.

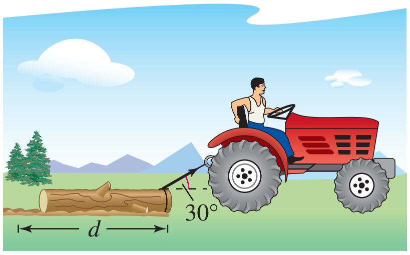
d	0°	1°	2°	3°	4°	5°
Force						

d	6°	7°	8°	9°	10°
Force					

(c) Find the force perpendicular to the hill when $d = 5°$.

62. ***Braking Load*** A sport utility vehicle with a gross weight of 5400 pounds is parked on a slope of 10°. Assume that the only force to overcome is the force of gravity. Find the force required to keep the vehicle from rolling down the hill. Find the force perpendicular to the hill.

63. ***Work*** A tractor pulls a log d meters and the tension in the cable connecting the tractor and log is approximately 1600 kilograms (15,691 newtons). The direction of the force is 30° above the horizontal (see figure).



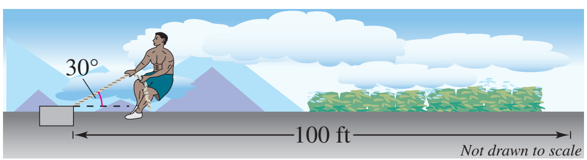
(a) Find the work done in terms of the distance d.

(b) Use a graphing utility to complete the table.

d	0	200	400	800
Work				

64. ***Work*** A force of 45 pounds in the direction of 30° above the horizontal is required to slide a table across a floor. Find the work done if the table is dragged 20 feet.

65. ***Work*** One of the events in a local strongman contest is to pull a cement block 100 feet. If a force of 250 pounds was used to pull the block at an angle of 30° with the horizontal, find the work done in pulling the block.



66. ***Work*** A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in pulling the wagon 50 feet.

67. ***Work*** A toy wagon is pulled by exerting a force of 20 pounds on a handle that makes a 25° angle with the horizontal. Find the work done in pulling the wagon 40 feet.

68. ***Work*** A mover exerts a horizontal force of 25 pounds on a crate as it is pushed up a ramp that is 12 feet long and inclined at an angle of 20° above the horizontal. Find the work done in pushing the crate up the ramp.

Synthesis

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The vectors $\mathbf{u} = \langle 0, 0 \rangle$ and $\mathbf{v} = \langle -12, 6 \rangle$ are orthogonal.

70. The work W done by a constant force $\mathbf{F}$ acting along the line of motion of an object is represented by a vector.

71. If $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and $\mathbf{v} = \langle \sin\theta, -\cos\theta \rangle$, are $\mathbf{u}$ and $\mathbf{v}$ orthogonal, parallel, or neither? Explain.

72. ***Think About It*** What is known about θ, the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, if each of the following is true?

(a) $\mathbf{u} \cdot \mathbf{v} = 0$ (b) $\mathbf{u} \cdot \mathbf{v} > 0$ (c) $\mathbf{u} \cdot \mathbf{v} < 0$

73. ***Think About It*** What can be said about the vectors $\mathbf{u}$ and $\mathbf{v}$ under each condition?

(a) The projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{u}$.

(b) The projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{0}$.

74. ***Proof*** Use vectors to prove that the diagonals of a rhombus are perpendicular.

75. ***Proof*** Prove the following.

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

76. ***Proof*** Prove that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $c\mathbf{v} + d\mathbf{w}$ for any scalars c and d.

77. ***Proof*** Prove that if $\mathbf{u}$ is a unit vector and θ is the angle between $\mathbf{u}$ and $\mathbf{i}$, then $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

78. ***Proof*** Prove that if $\mathbf{u}$ is a unit vector and θ is the angle between $\mathbf{u}$ and $\mathbf{j}$, then

$$\mathbf{u} = \cos\left(\frac{\pi}{2} - \theta\right)\mathbf{i} + \sin\left(\frac{\pi}{2} - \theta\right)\mathbf{j}.$$

Skills Review

In Exercises 79–82, describe how the graph of g is related to the graph of f.

79. $g(x) = f(x - 4)$

80. $g(x) = -f(x)$

81. $g(x) = f(x) + 6$

82. $g(x) = f(2x)$

In Exercises 83–90, perform the operation and write the result in standard form.

83. $\sqrt{-4} - 1$

84. $\sqrt{-8} + 5$

85. $3i(4 - 5i)$

86. $-2i(1 + 6i)$

87. $(1 + 3i)(1 - 3i)$

88. $(7 - 4i)(7 + 4i)$

89. $\dfrac{3}{1 + i} + \dfrac{2}{2 - 3i}$

90. $\dfrac{6}{4 - i} - \dfrac{3}{1 + i}$

In Exercises 91–94, plot the complex number in the complex plane.

91. $-2i$

92. $3i$

93. $1 + 8i$

94. $9 - 7i$

6.5 Trigonometric Form of a Complex Number

The Complex Plane

Recall from Section 2.4 that you can represent a complex number $z = a + bi$ as the point (a, b) in a coordinate plane (the complex plane). The horizontal axis is called the real axis and the vertical axis is called the imaginary axis, as shown in Figure 6.45.

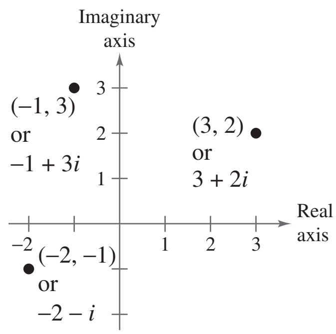


Figure 6.45

The **absolute value of a complex number** $a + bi$ is defined as the distance between the origin $(0, 0)$ and the point (a, b).

> **Definition of the Absolute Value of a Complex Number**
>
> The **absolute value** of the complex number $z = a + bi$ is given by
>
> $$|a + bi| = \sqrt{a^2 + b^2}.$$

If the complex number $a + bi$ is a real number (that is, if $b = 0$), then this definition agrees with that given for the absolute value of a real number

$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|.$$

Example 1 Finding the Absolute Value of a Complex Number

Plot $z = -2 + 5i$ and find its absolute value.

Solution

The number is plotted in Figure 6.46. It has an absolute value of

$$|z| = \sqrt{(-2)^2 + 5^2}$$
$$= \sqrt{29}.$$

CHECKPOINT Now try Exercise 5.

What you should learn

- Find absolute values of complex numbers.
- Write trigonometric forms of complex numbers.
- Multiply and divide complex numbers written in trigonometric form.
- Use DeMoivre's Theorem to find powers of complex numbers.
- Find *n*th roots of complex numbers.

Why you should learn it

You can use the trigonometric form of a complex number to perform operations with complex numbers. For instance, in Exercises 141–148 on page 459, you can use the trigonometric form of a complex number to help you solve polynomial equations.

Prerequisite Skills

To review complex numbers and the complex plane, see Section 2.4.

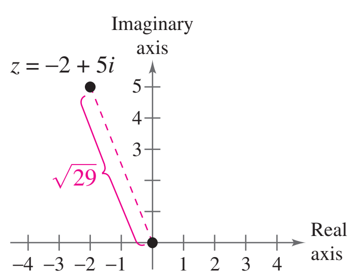


Figure 6.46

Trigonometric Form of a Complex Number

In Section 2.4 you learned how to add, subtract, multiply, and divide complex numbers. To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in trigonometric form. In Figure 6.47, consider the nonzero complex number $a + bi$. By letting θ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point (a, b), you can write

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r\cos\theta) + (r\sin\theta)i$$

from which you can obtain the **trigonometric form of a complex number.**

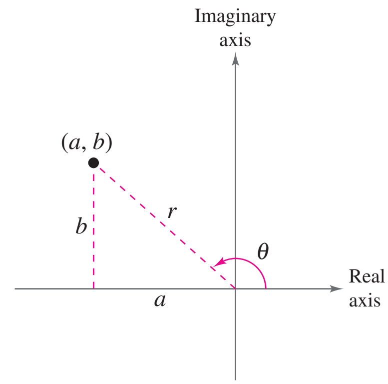


Figure 6.47

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is given by

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

The trigonometric form of a complex number is also called the *polar form*. Because there are infinitely many choices for θ, the trigonometric form of a complex number is not unique. Normally, θ is restricted to the interval $0 \le \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.

Example 2 Writing a Complex Number in Trigonometric Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

Solution

The absolute value of z is

$$r = \left|-2 - 2\sqrt{3}i\right| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

and the angle θ is given by

$$\tan\theta = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, choose θ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

See Figure 6.48.

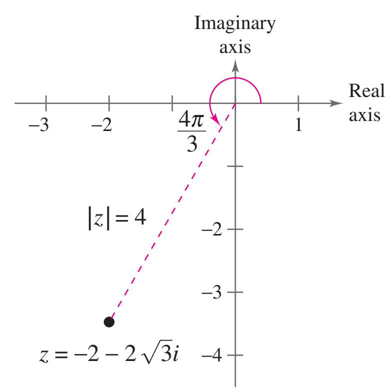


Figure 6.48

CHECKPOINT Now try Exercise 19.

Example 3 Writing a Complex Number in Standard Form

Write the complex number in standard form $a + bi$.

$$z = \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Solution

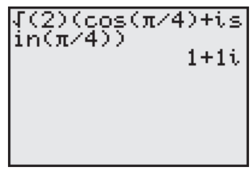
Because $\cos(-\pi/3) = 1/2$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can write

$$\begin{aligned} z &= \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \\ &= \sqrt{8}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \\ &= 2\sqrt{2}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = \sqrt{2} - \sqrt{6}i. \end{aligned}$$

CHECKPOINT Now try Exercise 37.

TECHNOLOGY TIP

A graphing utility can be used to convert a complex number in trigonometric form to standard form. For instance, enter the complex number $\sqrt{2}(\cos \pi/4 + i \sin \pi/4)$ in your graphing utility and press ENTER. You should obtain the standard form $1 + i$, as shown below.



Multiplication and Division of Complex Numbers

The trigonometric form adapts nicely to multiplication and division of complex numbers. Suppose you are given two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

The product of z_1 and z_2 is

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]. \end{aligned}$$

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following rule. The second part is left for you to verify (see Exercise 158).

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

Note that this rule says that to *multiply* two complex numbers you multiply moduli and add arguments, whereas to *divide* two complex numbers you divide moduli and subtract arguments.

Example 4 Multiplying Complex Numbers in Trigonometric Form

Find the product z_1z_2 of the complex numbers.

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad z_2 = 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Solution

$$\begin{aligned}
z_1z_2 &= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \cdot 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) \\
&= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] \\
&= 16\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) \\
&= 16\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
&= 16[0 + i(1)] = 16i
\end{aligned}$$

You can check this result by first converting to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically, as in Section 2.4.

$$\begin{aligned}
z_1z_2 &= \left(-1 + \sqrt{3}i\right)\left(4\sqrt{3} - 4i\right) \\
&= -4\sqrt{3} + 4i + 12i + 4\sqrt{3} = 16i
\end{aligned}$$

CHECKPOINT Now try Exercise 55.

TECHNOLOGY TIP

Some graphing utilities can multiply and divide complex numbers in trigonometric form. If you have access to such a graphing utility, use it to find z_1z_2 and z_1/z_2 in Examples 4 and 5.

Example 5 Dividing Complex Numbers in Trigonometric Form

Find the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24(\cos 300° + i\sin 300°) \qquad z_2 = 8(\cos 75° + i\sin 75°)$$

Solution

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{24(\cos 300° + i\sin 300°)}{8(\cos 75° + i\sin 75°)} \\
&= \frac{24}{8}[\cos(300° - 75°) + i\sin(300° - 75°)] \\
&= 3(\cos 225° + i\sin 225°) \\
&= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right] = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i
\end{aligned}$$

CHECKPOINT Now try Exercise 61.

Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$\begin{aligned}
z &= r(\cos\theta + i\sin\theta)\\
z^2 &= r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)\\
z^3 &= r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)\\
z^4 &= r^4(\cos 4\theta + i\sin 4\theta)\\
z^5 &= r^5(\cos 5\theta + i\sin 5\theta)\\
&\vdots
\end{aligned}$$

This pattern leads to **DeMoivre's Theorem,** which is named after the French mathematician Abraham DeMoivre (1667–1754).

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then

$$\begin{aligned}
z^n &= [r(\cos\theta + i\sin\theta)]^n\\
&= r^n(\cos n\theta + i\sin n\theta).
\end{aligned}$$

Exploration

Plot the numbers i, i^2, i^3, i^4, and i^5 in the complex plane. Write each number in trigonometric form and describe what happens to the angle θ as you form higher powers of i^n.

Example 6 Finding Powers of a Complex Number

Use DeMoivre's Theorem to find $\left(-1 + \sqrt{3}i\right)^{12}$.

Solution

First convert the complex number to trigonometric form using

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2 \quad\text{and}\quad \theta = \arctan\frac{\sqrt{3}}{-1} = -\frac{\pi}{3}.$$

So, the trigonometric form is

$$-1 + \sqrt{3}i = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned}
\left(-1 + \sqrt{3}i\right)^{12} &= \left(2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]\right)^{12}\\
&= 2^{12}\left[\cos\left(-\frac{12\pi}{3}\right) + i\sin\left(-\frac{12\pi}{3}\right)\right]\\
&= 4096[\cos(-4\pi) + i\sin(-4\pi)]\\
&= 4096(1 + 0) = 4096.
\end{aligned}$$

CHECKPOINT Now try Exercise 91.

Roots of Complex Numbers

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. So, an equation such as $x^6 = 1$ has six solutions, and in this particular case you can find the six solutions by factoring and using the Quadratic Formula.

$$x^6 - 1 = 0$$
$$(x^3 - 1)(x^3 + 1) = 0$$
$$(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = 0$$

Consequently, the solutions are

$$x = \pm 1, \qquad x = \frac{-1 \pm \sqrt{3}i}{2}, \qquad \text{and} \qquad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is a sixth root of 1. In general, the ***n*th root of a complex number** is defined as follows.

Definition of an nth Root of a Complex Number

The complex number $u = a + bi$ is an ***n*th root** of the complex number z if

$$z = u^n = (a + bi)^n.$$

To find a formula for an nth root of a complex number, let u be an nth root of z, where

$$u = s(\cos \beta + i \sin \beta) \qquad \text{and} \qquad z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by r, you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \qquad \text{and} \qquad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of 2π, these last two equations have solutions if and only if the angles differ by a multiple of 2π. Consequently, there must exist an integer k such that

$$n\beta = \theta + 2\pi k$$
$$\beta = \frac{\theta + 2\pi k}{n}.$$

By substituting this value of β into the trigonometric form of u, you get the result stated in the theorem on the following page.

Exploration

The nth roots of a complex number are useful for solving some polynomial equations. For instance, explain how you can use DeMoivre's Theorem to solve the polynomial equation

$$x^4 + 16 = 0.$$

[*Hint:* Write -16 as

$16(\cos \pi + i \sin \pi)$.]

*n*th Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

When $k > n - 1$ the roots begin to repeat. For instance, if $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

is coterminal with θ/n, which is also obtained when $k = 0$.

The formula for the nth roots of a complex number z has a nice geometrical interpretation, as shown in Figure 6.49. Note that because the nth roots of z all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive nth roots have arguments that differ by $2\pi/n$, the n roots are equally spaced around the circle.

You have already found the sixth roots of 1 by factoring and by using the Quadratic Formula. Example 7 shows how you can solve the same problem with the formula for nth roots.

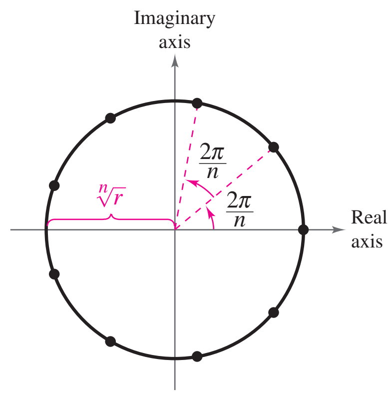


Figure 6.49

Example 7 Finding the *n*th Roots of a Real Number

Find all the sixth roots of 1.

Solution

First write 1 in the trigonometric form $1 = 1(\cos 0 + i\sin 0)$. Then, by the nth root formula with $n = 6$ and $r = 1$, the roots have the form

$$\sqrt[6]{1}\left(\cos\frac{0 + 2\pi k}{6} + i\sin\frac{0 + 2\pi k}{6}\right) = \cos\frac{\pi k}{3} + i\sin\frac{\pi k}{3}.$$

So, for $k = 0, 1, 2, 3, 4$, and 5, the sixth roots are as follows. (See Figure 6.50.)

$$\cos 0 + i\sin 0 = 1$$

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Incremented by $\frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos\pi + i\sin\pi = -1$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

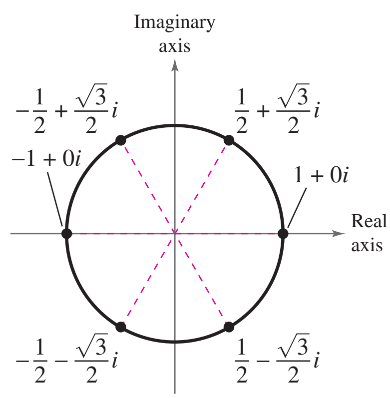


Figure 6.50

CHECKPOINT Now try Exercise 135.

In Figure 6.50, notice that the roots obtained in Example 7 all have a magnitude of 1 and are equally spaced around the unit circle. Also notice that the complex roots occur in conjugate pairs, as discussed in Section 2.5. The n distinct nth roots of 1 are called the ***n*th roots of unity.**

Example 8 Finding the nth Roots of a Complex Number

Find the three cube roots of $z = -2 + 2i$.

Solution

The absolute value of z is

$$r = |-2 + 2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

and the angle θ is given by

$$\tan\theta = \frac{b}{a} = \frac{2}{-2} = -1.$$

Because z lies in Quadrant II, the trigonometric form of z is

$$\begin{aligned} z &= -2 + 2i \\ &= \sqrt{8}(\cos 135^\circ + i \sin 135^\circ). \end{aligned}$$

By the formula for nth roots, the cube roots have the form

$$\sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ k}{3} + i\sin\frac{135^\circ + 360^\circ k}{3}\right).$$

Finally, for $k = 0, 1,$ and 2, you obtain the roots

$$\begin{aligned} &\sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(0)}{3} + i\sin\frac{135^\circ + 360^\circ(0)}{3}\right) \\ &= \sqrt{2}(\cos 45^\circ + i\sin 45^\circ) \\ &= 1 + i \end{aligned}$$

$$\begin{aligned} &\sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(1)}{3} + i\sin\frac{135^\circ + 360^\circ(1)}{3}\right) \\ &= \sqrt{2}(\cos 165^\circ + i\sin 165^\circ) \\ &\approx -1.3660 + 0.3660i \end{aligned}$$

$$\begin{aligned} &\sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(2)}{3} + i\sin\frac{135^\circ + 360^\circ(2)}{3}\right) \\ &= \sqrt{2}(\cos 285^\circ + i\sin 285^\circ) \\ &\approx 0.3660 - 1.3660i. \end{aligned}$$

See Figure 6.51.

CHECKPOINT Now try Exercise 139.

Exploration

Use a graphing utility set in *parametric* and *radian* modes to display the graphs of

X1T = cos T

and

Y1T = sin T.

Set the viewing window so that $-1.5 \le X \le 1.5$ and $-1 \le Y \le 1$. Then, using $0 \le T \le 2\pi$, set the "Tstep" to $2\pi/n$ for various values of n. Explain how the graphing utility can be used to obtain the nth roots of unity.

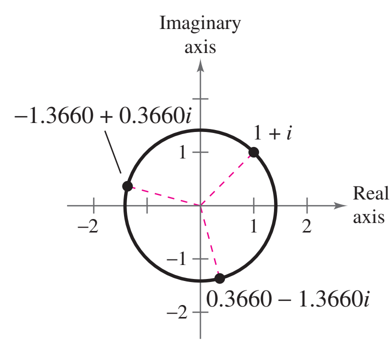


Figure 6.51

6.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

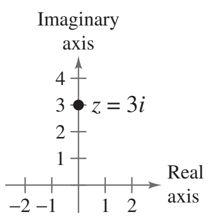
Fill in the blanks.

1. The _______ of a complex number $a + bi$ is the distance between the origin $(0, 0)$ and the point (a, b).
2. The _______ of a complex number $z = a + bi$ is given by $z = r(\cos\theta + i\sin\theta)$, where r is the _______ of z and θ is the _______ of z.
3. _______ Theorem states that if $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
4. The complex number $u = a + bi$ is an _______ of the complex number z if $z = u^n = (a + bi)^n$.

In Exercises 1–8, plot the complex number and find its absolute value.

1. $6i$
2. $-2i$
3. -4
4. 7
5. $-4 + 4i$
6. $-5 - 12i$
7. $3 + 6i$
8. $10 - 3i$

In Exercises 9–16, write the complex number in trigonometric form without using a calculator.

9.


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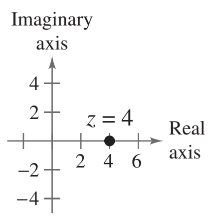

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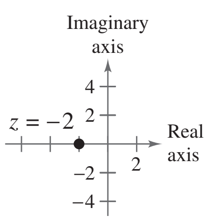

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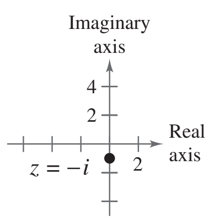

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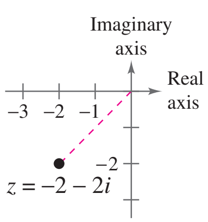

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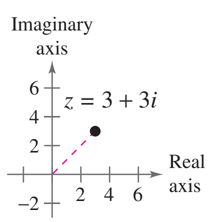

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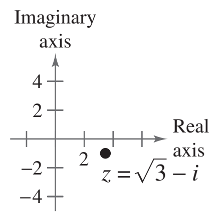

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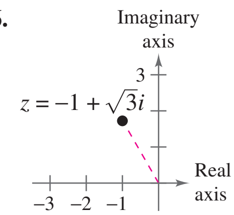


In Exercises 17–36, represent the complex number graphically, and find the trigonometric form of the number.

17. $5 - 5i$
18. $2 + 2i$
19. $\sqrt{3} + i$
20. $-1 - \sqrt{3}i$
21. $-2(1 + \sqrt{3}i)$
22. $\frac{5}{2}(\sqrt{3} - i)$
23. $-8i$
24. $4i$
25. $-7 + 4i$
26. $5 - i$
27. 3
28. 6
29. $3 + \sqrt{3}i$
30. $2\sqrt{2} - i$
31. $-1 - 2i$
32. $1 + 3i$
33. $5 + 2i$
34. $-3 + i$
35. $3\sqrt{2} - 7i$
36. $-8 - 5\sqrt{3}i$

In Exercises 37–48, represent the complex number graphically, and find the standard form of the number.

37. $2(\cos 120° + i\sin 120°)$
38. $5(\cos 135° + i\sin 135°)$
39. $\frac{3}{2}(\cos 330° + i\sin 330°)$
40. $\frac{3}{4}(\cos 315° + i\sin 315°)$
41. $3.75\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

42. $1.5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

43. $6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

44. $8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

45. $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

46. $9(\cos 0 + i\sin 0)$

47. $3[\cos(18°\,45') + i\sin(18°\,45')]$

48. $6[\cos(230°\,30') + i\sin(230°\,30')]$

In Exercises 49–52, use a graphing utility to represent the complex number in standard form.

49. $5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$

50. $12\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

51. $9(\cos 58° + i\sin 58°)$

52. $4(\cos 216.5° + i\sin 216.5°)$

In Exercises 53 and 54, represent the powers z, z^2, z^3, and z^4 graphically. Describe the pattern.

53. $z = \frac{\sqrt{2}}{2}(1 + i)$

54. $z = \frac{1}{2}\left(1 + \sqrt{3}i\right)$

In Exercises 55–66, perform the operation and leave the result in trigonometric form.

55. $\left[3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$

56. $\left[\frac{3}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]\left[6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]$

57. $\left[\frac{5}{3}(\cos 140° + i\sin 140°)\right]\left[\frac{2}{3}(\cos 60° + i\sin 60°)\right]$

58. $\left[\frac{1}{2}(\cos 115° + i\sin 115°)\right]\left[\frac{4}{5}(\cos 300° + i\sin 300°)\right]$

59. $\left[\frac{11}{20}(\cos 290° + i\sin 290°)\right]\left[\frac{2}{5}(\cos 200° + i\sin 200°)\right]$

60. $(\cos 5° + i\sin 5°)(\cos 20° + i\sin 20°)$

61. $\dfrac{\cos 50° + i\sin 50°}{\cos 20° + i\sin 20°}$

62. $\dfrac{5(\cos 4.3 + i\sin 4.3)}{4(\cos 2.1 + i\sin 2.1)}$

63. $\dfrac{2(\cos 120° + i\sin 120°)}{4(\cos 40° + i\sin 40°)}$

64. $\dfrac{\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)}{\cos\pi + i\sin\pi}$

65. $\dfrac{18(\cos 54° + i\sin 54°)}{3(\cos 102° + i\sin 102°)}$

66. $\dfrac{9(\cos 20° + i\sin 20°)}{5(\cos 75° + i\sin 75°)}$

In Exercises 67–82, (a) write the trigonometric forms of the complex numbers, (b) perform the indicated operation using the trigonometric forms, and (c) perform the indicated operation using the standard forms and check your result with that of part (b).

67. $(2 - 2i)(1 + i)$

68. $(3 - 3i)(1 - i)$

69. $(2 + 2i)(1 - i)$

70. $\left(\sqrt{3} + i\right)(1 + i)$

71. $-2i(1 + i)$

72. $3i(1 + i)$

73. $-2i(\sqrt{3} - i)$

74. $-i(1 + \sqrt{3}i)$

75. $2(1 - i)$

76. $-4(1 + i)$

77. $\dfrac{3 + 3i}{1 - \sqrt{3}i}$

78. $\dfrac{2 + 2i}{1 + \sqrt{3}i}$

79. $\dfrac{5}{2 + 2i}$

80. $\dfrac{2}{\sqrt{3} - i}$

81. $\dfrac{4i}{-1 + i}$

82. $\dfrac{2i}{1 - \sqrt{3}i}$

In Exercises 83–90, sketch the graph of all complex numbers z satisfying the given condition.

83. $|z| = 2$

84. $|z| = 5$

85. $|z| = 4$

86. $|z| = 6$

87. $\theta = \dfrac{\pi}{6}$

88. $\theta = \dfrac{\pi}{4}$

89. $\theta = \dfrac{5\pi}{6}$

90. $\theta = \dfrac{2\pi}{3}$

In Exercises 91–110, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

91. $(1 + i)^3$

92. $(2 + 2i)^6$

93. $(-1 + i)^{10}$

94. $(1 - i)^8$

95. $2(\sqrt{3} + i)^5$

96. $4(1 - \sqrt{3}i)^3$

97. $[5(\cos 20° + i \sin 20°)]^3$

98. $[3(\cos 150° + i \sin 150°)]^4$

99. $\left(\cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4}\right)^{10}$

100. $\left[2\left(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)\right]^{12}$

101. $[2(\cos 1.25 + i \sin 1.25)]^4$

102. $[4(\cos 2.8 + i \sin 2.8)]^5$

103. $[2(\cos \pi + i \sin \pi)]^8$

104. $(\cos 0 + i \sin 0)^{20}$

105. $(3 - 2i)^5$

106. $(\sqrt{5} - 4i)^4$

107. $[4(\cos 10° + i \sin 10°)]^6$

108. $[3(\cos 15° + i \sin 15°)]^4$

109. $\left[3\left(\cos \dfrac{\pi}{8} + i \sin \dfrac{\pi}{8}\right)\right]^2$

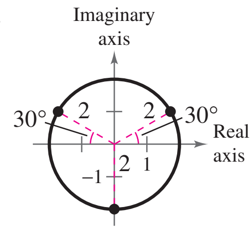
110. $\left[2\left(\cos \dfrac{\pi}{10} + i \sin \dfrac{\pi}{10}\right)\right]^5$

111. Show that $-\frac{1}{2}(1 + \sqrt{3}i)$ is a sixth root of 1.

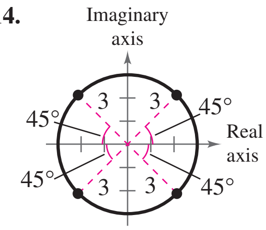
112. Show that $2^{-1/4}(1 - i)$ is a fourth root of -2.

Graphical Reasoning **In Exercises 113–116, use the graph of the roots of a complex number. (a) Write each of the roots in trigonometric form. (b) Identify the complex number whose roots are given. (c) Use a graphing utility to verify the results of part (b).**

113.



114.



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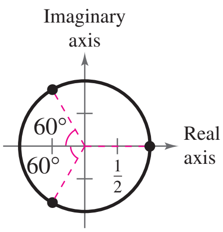

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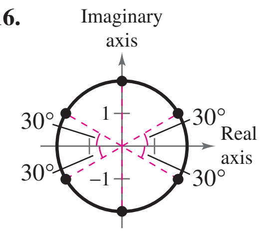


In Exercises 117–124, find the square roots of the complex number.

117. $2i$

118. $5i$

119. $-3i$

120. $-6i$

121. $2 - 2i$

122. $2 + 2i$

123. $1 + \sqrt{3}i$

124. $1 - \sqrt{3}i$

In Exercises 125–140, (a) use the theorem on page 454 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

125. Square roots of $5(\cos 120° + i \sin 120°)$

126. Square roots of $16(\cos 60° + i \sin 60°)$

127. Fourth roots of $16\left(\cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3}\right)$

128. Fifth roots of $32\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$

129. Cube roots of $-27i$

130. Fourth roots of $625i$

131. Cube roots of $-\frac{125}{2}(1 + \sqrt{3}i)$

132. Cube roots of $-4\sqrt{2}(1 - i)$

133. Cube roots of $64i$

134. Fourth roots of i

135. Fifth roots of 1

136. Cube roots of 1000

137. Cube roots of -125

138. Fourth roots of -4

139. Fifth roots of $128(-1 + i)$

140. Sixth roots of $729i$

In Exercises 141–148, use the theorem on page 454 to find all the solutions of the equation, and represent the solutions graphically.

141. $x^4 - i = 0$

142. $x^3 + 27 = 0$

143. $x^5 + 243 = 0$

144. $x^4 - 81 = 0$

145. $x^4 + 16i = 0$

146. $x^6 - 64i = 0$

147. $x^3 - (1 - i) = 0$

148. $x^4 + (1 + i) = 0$

Electrical Engineering **In Exercises 149–154, use the formula to find the missing quantity for the given conditions. The formula**

$$E = I \cdot Z$$

where E represents voltage, I represents current, and Z represents impedance (a measure of opposition to a sinusoidal electric current), is used in electrical engineering. Each variable is a complex number.

149. $I = 10 + 2i$, $Z = 4 + 3i$

150. $I = 12 + 2i$, $Z = 3 + 5i$

151. $I = 2 + 4i$, $E = 5 + 5i$

152. $I = 10 + 2i$, $E = 4 + 5i$

153. $E = 12 + 24i$, $Z = 12 + 20i$

154. $E = 15 + 12i$, $Z = 25 + 24i$

Synthesis

True or False? **In Exercises 155–157, determine whether the statement is true or false. Justify your answer.**

155. $\frac{1}{2}(1 - \sqrt{3}i)$ is a ninth root of -1.

156. $\sqrt{3} + i$ is a solution of the equation $x^2 - 8i = 0$.

157. Geometrically, the nth roots of any complex number z are all equally spaced around the unit circle centered at the origin.

158. Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $z_2 \neq 0$, show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

159. Show that $\bar{z} = r[\cos(-\theta) + i\sin(-\theta)]$ is the complex conjugate of $z = r(\cos\theta + i\sin\theta)$.

160. Use the trigonometric forms of z and $\bar{z}$ in Exercise 159 to find (a) $z\bar{z}$ and (b) $z/\bar{z}$, $\bar{z} \neq 0$.

161. Show that the negative of $z = r(\cos\theta + i\sin\theta)$ is $-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)]$.

162. ***Writing*** The famous formula

$$e^{a+bi} = e^a(\cos b + i\sin b)$$

is called Euler's Formula, after the Swiss mathematician Leonhard Euler (1707–1783). This formula gives rise to the equation

$$e^{\pi i} + 1 = 0.$$

This equation relates the five most famous numbers in mathematics—0, 1, π, e, and i—in a single equation. Show how Euler's Formula can be used to derive this equation. Write a short paragraph summarizing your work.

Skills Review

Harmonic Motion **In Exercises 163–166, for the simple harmonic motion described by the trigonometric function, find the maximum displacement from equilibrium and the lowest possible positive value of t for which $d = 0$.**

163. $d = 16\cos\frac{\pi}{4}t$

164. $d = \frac{1}{16}\sin\frac{5\pi}{4}t$

165. $d = \frac{1}{8}\cos 12\pi t$

166. $d = \frac{1}{12}\sin 60\pi t$

In Exercises 167–170, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to verify your answers.

167. $2\cos(x + \pi) + 2\cos(x - \pi) = 0$

168. $\sin\left(x + \frac{3\pi}{2}\right) - \sin\left(x - \frac{3\pi}{2}\right) = 0$

169. $\sin\left(x - \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{3}\right) = \frac{3}{2}$

170. $\tan(x + \pi) - \cos\left(x + \frac{5\pi}{2}\right) = 0$

What Did You Learn?

Key Terms

oblique triangle, *p. 408*
directed line segment, *p. 424*
vector $\mathbf{v}$ in the plane, *p. 424*
standard position of a vector, *p. 425*
zero vector, *p. 425*
parallelogram law, *p. 426*
resultant, *p. 426*
standard unit vector, *p. 429*
horizontal and vertical components of $\mathbf{v}$, *p. 429*
linear combination, *p. 429*
direction angle, *p. 430*
vector components, *p. 441*
absolute value of a complex number, *p. 448*
trigonometric form of a complex number, *p. 449*
modulus, *p. 449*
argument, *p. 449*
DeMoivre's Theorem, *p. 452*
*n*th root of a complex number, *p. 453*
*n*th roots of unity, *p. 455*

Key Concepts

6.1 ■ Use the Law of Sines to solve oblique triangles

If ABC is a triangle with sides a, b, and c, then the Law of Sines is as follows.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

6.1 ■ Find areas of oblique triangles

To find the area of a triangle given two sides and their included angle, use one of the following formulas.

$$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

6.2 ■ Use the Law of Cosines to solve oblique triangles

1. $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
2. $b^2 = a^2 + c^2 - 2ac\cos B$ or $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
3. $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

6.2 ■ Use Heron's Area Formula to find areas of triangles

Given any triangle with sides of lengths a, b, and c, the area of the triangle is given by

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a + b + c)/2$.

6.3 ■ Use vectors in the plane

1. The component form of a vector $\mathbf{v}$ with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by $\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}$.
2. The magnitude (or length) of $\mathbf{v}$ is given by $\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}$.
3. The sum of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
4. The scalar multiple of k times $\mathbf{u}$ is $k\mathbf{u} = \langle ku_1, ku_2 \rangle$.
5. A unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is $\mathbf{u} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$.

6.4 ■ Use the dot product of two vectors

1. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$.
2. If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.
3. The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
4. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$.
5. The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ or $W = \mathbf{F} \cdot \overrightarrow{PQ}$.

6.5 ■ Find *n*th roots of complex numbers

For a positive integer n, the complex number

$$z = r(\cos\theta + i\sin\theta)$$

has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

6.1 In Exercises 1–12, use the Law of Sines to solve the triangle. If two solutions exist, find both.

1. $A = 32°, \quad B = 50°, \quad a = 16$

2. $A = 38°, \quad B = 58°, \quad a = 12$

3. $B = 25°, \quad C = 105°, \quad c = 25$

4. $B = 20°, \quad C = 115°, \quad c = 30$

5. $A = 60°15', \quad B = 45°30', \quad b = 4.8$

6. $A = 82°45', \quad B = 28°45', \quad b = 40.2$

7. $A = 75°, \quad a = 2.5, \quad b = 16.5$

8. $A = 15°, \quad a = 5, \quad b = 10$

9. $B = 115°, \quad a = 9, \quad b = 14.5$

10. $B = 150°, \quad a = 64, \quad b = 10$

11. $C = 50°, \quad a = 25, \quad c = 22$

12. $B = 25°, \quad a = 6.2, \quad b = 4$

In Exercises 13–16, find the area of the triangle having the indicated angle and sides.

13. $A = 27°, \quad b = 5, \quad c = 8$

14. $B = 80°, \quad a = 4, \quad c = 8$

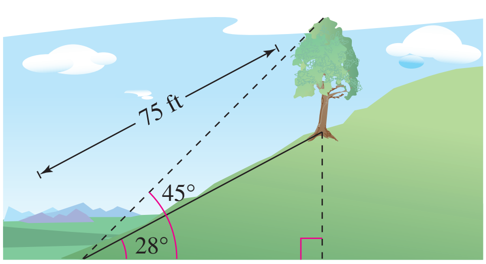
15. $C = 122°, \quad b = 18, \quad a = 29$

16. $C = 100°, \quad a = 120, \quad b = 74$

17. ***Height*** From a distance of 50 meters, the angle of elevation to the top of a building is 17°. Approximate the height of the building.

18. ***Distance*** A family is traveling due west on a road that passes a famous landmark. At a given time the bearing to the landmark is N 62° W, and after the family travels 5 miles farther, the bearing is N 38° W. What is the closest the family will come to the landmark while on the road?

19. ***Height*** A tree stands on a hillside of slope 28° from the horizontal. From a point 75 feet down the hill, the angle of elevation to the top of the tree is 45° (see figure). Find the height of the tree.



20. ***Width*** A surveyor finds that a tree on the opposite bank of a river has a bearing of N 22° 30′ E from a certain point and a bearing of N 15° W from a point 400 feet downstream. Find the width of the river.

6.2 In Exercises 21–32, use the Law of Cosines to solve the triangle.

21. $a = 18, \quad b = 12, \quad c = 15$

22. $a = 10, \quad b = 12, \quad c = 16$

23. $a = 9, \quad b = 12, \quad c = 20$

24. $a = 7, \quad b = 15, \quad c = 19$

25. $a = 6.5, \quad b = 10.2, \quad c = 16$

26. $a = 6.2, \quad b = 6.4, \quad c = 2.1$

27. $C = 65°, \quad a = 25, \quad b = 12$

28. $B = 48°, \quad a = 18, \quad c = 12$

29. $B = 110°, \quad a = 4, \quad c = 4$

30. $B = 150°, \quad a = 10, \quad c = 20$

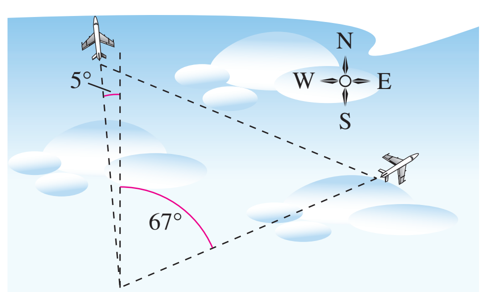
31. $B = 55°30', \quad a = 12.4, \quad c = 18.5$

32. $B = 85°15', \quad a = 24.2, \quad c = 28.2$

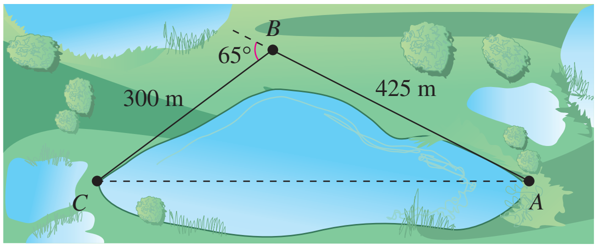
33. ***Geometry*** The lengths of the diagonals of a parallelogram are 10 feet and 16 feet. Find the lengths of the sides of the parallelogram if the diagonals intersect at an angle of 28°.

34. ***Geometry*** The lengths of the diagonals of a parallelogram are 30 meters and 40 meters. Find the lengths of the sides of the parallelogram if the diagonals intersect at an angle of 34°.

35. ***Navigation*** Two planes leave Washington, D.C.'s Dulles International Airport at approximately the same time. One is flying at 425 miles per hour at a bearing of 355°, and the other is flying at 530 miles per hour at a bearing of 67° (see figure). Determine the distance between the planes after they have flown for 2 hours.



36. ***Surveying*** To approximate the length of a marsh, a surveyor walks 425 meters from point A to point B. The surveyor then turns 65° and walks 300 meters to point C (see figure). Approximate the length AC of the marsh.



In Exercises 37–40, use Heron's Area Formula to find the area of the triangle with the given side lengths.

37. $a = 4, \quad b = 5, \quad c = 7$

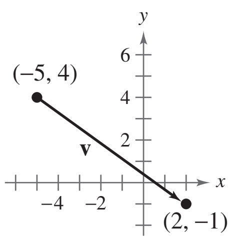
38. $a = 15, \quad b = 8, \quad c = 10$

39. $a = 64.8, \quad b = 49.2, \quad c = 24.1$

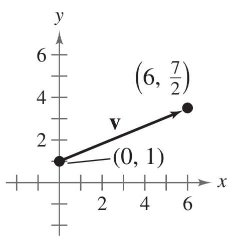
40. $a = 8.55, \quad b = 5.14, \quad c = 12.73$

6.3 In Exercises 41–46, find the component form of the vector v satisfying the given conditions.

41.



42.



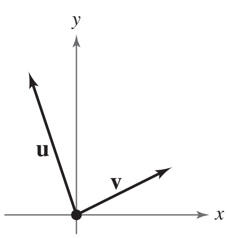
43. Initial point: (0, 10); terminal point: (7, 3)

44. Initial point: (1, 5); terminal point: (15, 9)

45. $\|\mathbf{v}\| = 8, \quad \theta = 120°$

46. $\|\mathbf{v}\| = \frac{1}{2}, \quad \theta = 225°$

In Exercises 47–52, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



47. $2\mathbf{u}$

48. $-\frac{1}{2}\mathbf{v}$

49. $2\mathbf{u} + \mathbf{v}$

50. $\mathbf{u} + 2\mathbf{v}$

51. $\mathbf{u} - 2\mathbf{v}$

52. $\mathbf{v} - 2\mathbf{u}$

In Exercises 53–60, find (a) u + v, (b) u − v, (c) 3u, and (d) 2v + 5u.

53. $\mathbf{u} = \langle -1, -3\rangle, \mathbf{v} = \langle -3, 6\rangle$

54. $\mathbf{u} = \langle 4, 5\rangle, \mathbf{v} = \langle 0, -1\rangle$

55. $\mathbf{u} = \langle -5, 2\rangle, \mathbf{v} = \langle 4, 4\rangle$

56. $\mathbf{u} = \langle 1, -8\rangle, \mathbf{v} = \langle 3, -2\rangle$

57. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}, \mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$

58. $\mathbf{u} = -6\mathbf{j}, \mathbf{v} = \mathbf{i} + \mathbf{j}$

59. $\mathbf{u} = 4\mathbf{i}, \quad \mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

60. $\mathbf{u} = -7\mathbf{i} - 3\mathbf{j}, \quad \mathbf{v} = 4\mathbf{i} - \mathbf{j}$

In Exercises 61–64, find the component form of w and sketch the specified vector operations geometrically, where u = 6i − 5j and v = 10i + 3j.

61. $\mathbf{w} = 3\mathbf{v}$

62. $\mathbf{w} = \frac{1}{2}\mathbf{v}$

63. $\mathbf{w} = 4\mathbf{u} + 5\mathbf{v}$

64. $\mathbf{w} = 3\mathbf{v} - 2\mathbf{u}$

In Exercises 65–68, find a unit vector in the direction of the given vector.

65. $\mathbf{u} = \langle 0, -6\rangle$

66. $\mathbf{v} = \langle -12, -5\rangle$

67. $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j}$

68. $\mathbf{w} = -7\mathbf{i}$

In Exercises 69 and 70, write a linear combination of the standard unit vectors i and j for the given initial and terminal points.

69. Initial point: $(-8, 3)$
Terminal point: $(1, -5)$

70. Initial point: $(2, -3.2)$
Terminal point: $(-6.4, 10.8)$

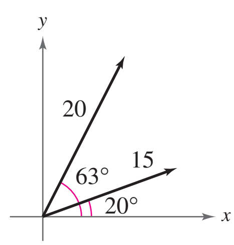
In Exercises 71 and 72, write the vector v in the form $\|\mathbf{v}\|[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}]$.

71. $\mathbf{v} = -10\mathbf{i} + 10\mathbf{j}$

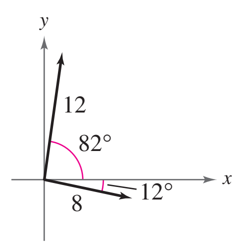
72. $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

In Exercises 73 and 74, graph the vectors and the resultant of the vectors. Find the magnitude and direction of the resultant.

73.



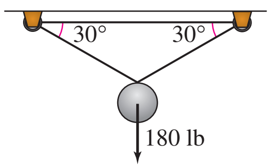
74.



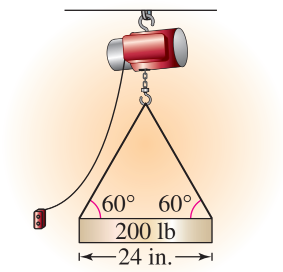
75. ***Resultant Force*** Three forces with magnitudes of 250 pounds, 100 pounds, and 200 pounds act on an object at angles of 60°, 150°, and −90°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

76. ***Resultant Force*** Forces with magnitudes of 85 pounds and 50 pounds act on a single point. The angle between the forces is 15°. Describe the resultant force.

77. ***Tension*** A 180-pound weight is supported by two ropes, as shown in the figure. Find the tension in each rope.



78. ***Cable Tension*** In a manufacturing process, an electric hoist lifts 200-pound ingots. Find the tension in the supporting cables (see figure).



79. ***Navigation*** An airplane has an airspeed of 430 miles per hour at a bearing of 135°. The wind velocity is 35 miles per hour in the direction N 30° E. Find the resultant speed and direction of the plane.

80. ***Navigation*** An airplane has an airspeed of 724 kilometers per hour at a bearing of 30°. The wind velocity is from the west at 32 kilometers per hour. Find the resultant speed and direction of the plane.

6.4 In Exercises 81–84, find the dot product of u and v.

81. $\mathbf{u} = \langle 0, -2 \rangle$, $\mathbf{v} = \langle 1, 10 \rangle$

82. $\mathbf{u} = \langle -4, 5 \rangle$, $\mathbf{v} = \langle 3, -1 \rangle$

83. $\mathbf{u} = 6\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$

84. $\mathbf{u} = 8\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

In Exercises 85–88, use the vectors $\mathbf{u} = \langle -3, -4 \rangle$ and $\mathbf{v} = \langle 2, 1 \rangle$ to find the indicated quantity.

85. $\mathbf{u} \cdot \mathbf{u}$

86. $\|\mathbf{v}\| - 3$

87. $4\mathbf{u} \cdot \mathbf{v}$

88. $(\mathbf{u} \cdot \mathbf{v})\mathbf{u}$

In Exercises 89–92, find the angle θ between the vectors.

89. $\mathbf{u} = \langle 2\sqrt{2}, -4 \rangle, \quad \mathbf{v} = \langle -\sqrt{2}, 1 \rangle$

90. $\mathbf{u} = \langle 3, 1 \rangle, \quad \mathbf{v} = \langle 4, 5 \rangle$

91. $\mathbf{u} = \cos \dfrac{7\pi}{4}\mathbf{i} + \sin \dfrac{7\pi}{4}\mathbf{j}, \mathbf{v} = \cos \dfrac{5\pi}{6}\mathbf{i} + \sin \dfrac{5\pi}{6}\mathbf{j}$

92. $\mathbf{u} = \cos 45^\circ\mathbf{i} + \sin 45^\circ\mathbf{j}$
$\mathbf{v} = \cos 300^\circ\mathbf{i} + \sin 300^\circ\mathbf{j}$

In Exercises 93–96, graph the vectors and find the degree measure of the angle between the vectors.

93. $\mathbf{u} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$

94. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -3\mathbf{i} - \mathbf{j}$

95. $\mathbf{u} = 7\mathbf{i} - 5\mathbf{j}$, $\mathbf{v} = 10\mathbf{i} + 3\mathbf{j}$

96. $\mathbf{u} = -5.3\mathbf{i} + 2.8\mathbf{j}$, $\mathbf{v} = -8.1\mathbf{i} - 4\mathbf{j}$

In Exercises 97–100, determine whether u and v are orthogonal, parallel, or neither.

97. $\mathbf{u} = \langle 39, -12 \rangle$, $\mathbf{v} = \langle -26, 8 \rangle$

98. $\mathbf{u} = \langle 8, -4 \rangle$, $\mathbf{v} = \langle 5, 10 \rangle$

99. $\mathbf{u} = \langle 8, 5 \rangle$, $\mathbf{v} = \langle -2, 4 \rangle$

100. $\mathbf{u} = \langle -15, 51 \rangle$, $\mathbf{v} = \langle 20, -68 \rangle$

In Exercises 101–104, find the value of k so that the vectors u and v are orthogonal.

101. $\mathbf{u} = \mathbf{i} - k\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

102. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -\mathbf{i} - k\mathbf{j}$

103. $\mathbf{u} = k\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j}$

104. $\mathbf{u} = k\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j}$

In Exercises 105–108, find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

105. $\mathbf{u} = \langle -4, 3 \rangle, \quad \mathbf{v} = \langle -8, -2 \rangle$

106. $\mathbf{u} = \langle 5, 6 \rangle, \quad \mathbf{v} = \langle 10, 0 \rangle$

107. $\mathbf{u} = \langle 2, 7 \rangle, \quad \mathbf{v} = \langle 1, -1 \rangle$

108. $\mathbf{u} = \langle -3, 5 \rangle, \quad \mathbf{v} = \langle -5, 2 \rangle$

109. ***Work*** Determine the work done by a crane lifting an 18,000-pound truck 48 inches.

110. ***Braking Force*** A 500-pound motorcycle is headed up a hill inclined at 12°. What force is required to keep the motorcycle from rolling back down the hill when stopped at a red light?

6.5 In Exercises 111–114, plot the complex number and find its absolute value.

111. $-i$

112. $5i$

113. $7 - 5i$

114. $-3 + 9i$

In Exercises 115–120, write the complex number in trigonometric form without using a calculator.

115. $2 - 2i$

116. $-2 + 2i$

117. $-\sqrt{3} - i$

118. $-\sqrt{3} + i$

119. $-2i$

120. $4i$

In Exercises 121–124, perform the operation and leave the result in trigonometric form.

121. $\left[\frac{5}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]\left[4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]$

122. $\left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]\left[3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$

123. $\dfrac{20(\cos 320° + i\sin 320°)}{5(\cos 80° + i\sin 80°)}$

124. $\dfrac{3(\cos 230° + i\sin 230°)}{9(\cos 95° + i\sin 95°)}$

In Exercises 125–130, (a) write the trigonometric forms of the complex numbers, (b) perform the indicated operation using the trigonometric forms, and (c) perform the indicated operation using the standard forms and check your result with that of part (b).

125. $(2 - 2i)(3 + 3i)$

126. $(4 + 4i)(-1 - i)$

127. $-i(2 + 2i)$

128. $4i(1 - i)$

129. $\dfrac{3 - 3i}{2 + 2i}$

130. $\dfrac{-1 - i}{-2 - 2i}$

In Exercises 131–134, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

131. $\left[5\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]^4$

132. $\left[2\left(\cos\frac{4\pi}{15} + i\sin\frac{4\pi}{15}\right)\right]^5$

133. $(2 + 3i)^6$

134. $(1 - i)^8$

In Exercises 135–140, find the square roots of the complex number.

135. $-\sqrt{3} + i$

136. $\sqrt{3} - i$

137. $-2i$

138. $-5i$

139. $-2 - 2i$

140. $-2 + 2i$

In Exercises 141–144, (a) use the theorem on page 454 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

141. Sixth roots of $-729i$

142. Fourth roots of $256i$

143. Cube roots of 8

144. Fifth roots of -1024

In Exercises 145–148, use the theorem on page 454 to find all solutions of the equation, and represent the solutions graphically.

145. $x^4 + 256 = 0$

146. $x^5 - 32i = 0$

147. $x^3 + 8i = 0$

148. $x^4 + 81 = 0$

Synthesis

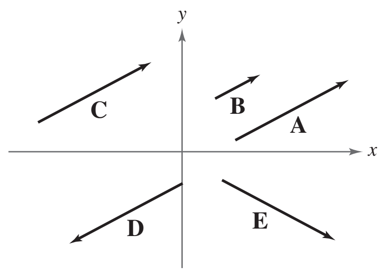
In Exercises 149 and 150, determine whether the statement is true or false. Justify your answer.

149. The Law of Sines is true if one of the angles in the triangle is a right angle.

150. When the Law of Sines is used, the solution is always unique.

151. What characterizes a vector in the plane?

152. Which vectors in the figure appear to be equivalent?



153. The figure shows z_1 and z_2. Describe $z_1 z_2$ and z_1/z_2.

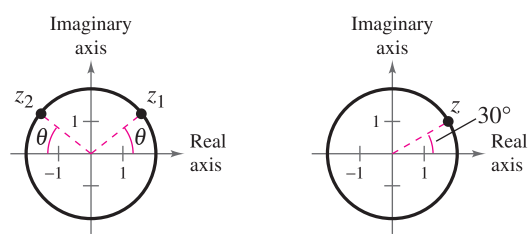


Figure for 153

Figure for 154

154. One of the fourth roots of a complex number z is shown in the graph.

(a) How many roots are not shown?

(b) Describe the other roots.

6 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–6, use the given information to solve the triangle. If two solutions exist, find both solutions.

1. $A = 36°,\ B = 98°,\ c = 16$

2. $a = 4,\ b = 8,\ c = 10$

3. $A = 35°,\ b = 8,\ c = 12$

4. $A = 25°,\ b = 28,\ a = 18$

5. $B = 130°,\ c = 10.1,\ b = 5.2$

6. $A = 150°,\ b = 4.8,\ a = 9.4$

7. Find the length of the pond shown at the right.

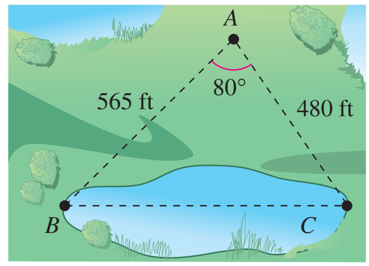


Figure for 7

8. A triangular parcel of land has borders of lengths 55 meters, 85 meters, and 100 meters. Find the area of the parcel of land.

9. Find the component form and magnitude of the vector $\mathbf{w}$ that has initial point $(-8, -12)$ and terminal point $(4, 1)$.

In Exercises 10–13, find (a) $2\mathbf{v} + \mathbf{u}$, (b) $\mathbf{u} - 3\mathbf{v}$, and (c) $5\mathbf{u} - \mathbf{v}$.

10. $\mathbf{u} = \langle 0, -4\rangle, \quad \mathbf{v} = \langle -2, -8\rangle$

11. $\mathbf{u} = \langle -2, -3\rangle, \quad \mathbf{v} = \langle -1, -10\rangle$

12. $\mathbf{u} = \mathbf{i} - \mathbf{j}, \quad \mathbf{v} = 6\mathbf{i} + 9\mathbf{j}$

13. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{v} = -\mathbf{i} - 2\mathbf{j}$

14. Find a unit vector in the direction of $\mathbf{v} = 7\mathbf{i} + 4\mathbf{j}$.

15. Find the component form of the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 12$, in the same direction as $\mathbf{u} = \langle 3, -5\rangle$.

16. Forces with magnitudes of 250 pounds and 130 pounds act on an object at angles of 45° and $-60°$, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

17. Find the dot product of $\mathbf{u} = \langle -9, 4\rangle$ and $\mathbf{v} = \langle 1, 3\rangle$.

18. Find the angle between the vectors $\mathbf{u} = 7\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -4\mathbf{j}$.

19. Are the vectors $\mathbf{u} = \langle 6, -4\rangle$ and $\mathbf{v} = \langle 2, -3\rangle$ orthogonal? Explain.

20. Find the projection of $\mathbf{u} = \langle 6, 7\rangle$ onto $\mathbf{v} = \langle -5, -1\rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors.

21. Write the complex number $z = -2 + 2i$ in trigonometric form.

22. Write the complex number $100(\cos 240° + i \sin 240°)$ in standard form.

In Exercises 23 and 24, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

23. $\left[3\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)\right]^8$

24. $(3 - 3i)^6$

25. Find the fourth roots of $128(1 + \sqrt{3}i)$.

26. Find all solutions of the equation $x^4 - 625i = 0$ and represent the solutions graphically.

4–6 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test to review the material in Chapters 4–6. After you are finished, check your work against the answers given in the back of the book.

1. Consider the angle $\theta = -150°$.
   (a) Sketch the angle in standard position.
   (b) Determine a coterminal angle in the interval $[0°, 360°)$.
   (c) Convert the angle to radian measure.
   (d) Find the reference angle θ'.
   (e) Find the exact values of the six trigonometric functions of θ.
2. Convert the angle $\theta = 2.55$ radians to degrees. Round your answer to one decimal place.
3. Find $\cos\theta$ if $\tan\theta = -\frac{12}{5}$ and $\sin\theta > 0$.

In Exercises 4–6, sketch the graph of the function by hand. (Include two full periods.) Use a graphing utility to verify your graph.

4. $f(x) = 3 - 2\sin \pi x$ 5. $f(x) = \tan 3x$ 6. $f(x) = \frac{1}{2}\sec(x + \pi)$

7. Find a, b, and c such that the graph of the function $h(x) = a\cos(bx + c)$ matches the graph in the figure at the right.

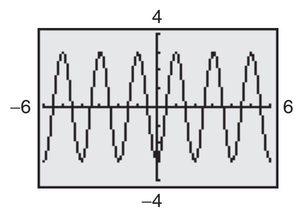


Figure for 7

In Exercises 8 and 9, find the exact value of the expression without using a calculator.

8. $\sin\left(\arctan \frac{3}{4}\right)$ 9. $\tan\left[\arcsin\left(-\frac{1}{2}\right)\right]$

10. Write an algebraic expression equivalent to $\sin(\arctan 2x)$.

11. Subtract and simplify: $\dfrac{\sin\theta - 1}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta - 1}$.

In Exercises 12–14, verify the identity.

12. $\cot^2\alpha(\sec^2\alpha - 1) = 1$ 13. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

14. $\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x)$

In Exercises 15 and 16, solve the equation.

15. $\sin^2 x + 2\sin x + 1 = 0$ 16. $3\tan\theta - \cot\theta = 0$

17. Approximate the solutions to the equation $\cos^2 x - 5\cos x - 1 = 0$ in the interval $[0, 2\pi)$.

In Exercises 18 and 19, use a graphing utility to graph the function and approximate its zeros in the interval $[0, 2\pi)$. If possible, find the exact values of the zeros algebraically.

18. $y = \dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} - 4$

19. $y = \tan^3 x - \tan^2 x + 3\tan x - 3$

20. Given that $\sin u = \frac{12}{13}$, $\cos v = \frac{3}{5}$, and angles u and v are both in Quadrant I, find $\tan(u - v)$.

21. If $\tan \theta = \dfrac{1}{2}$, find the exact value of $\tan 2\theta$, $0 < \theta < \dfrac{\pi}{2}$.

22. If $\tan \theta = \dfrac{4}{3}$, find the exact value of $\sin \dfrac{\theta}{2}$, $\pi < \theta < \dfrac{3\pi}{2}$.

23. Write $\cos 8x + \cos 4x$ as a product.

In Exercises 24–27, verify the identity.

24. $\tan x(1 - \sin^2 x) = \frac{1}{2} \sin 2x$

25. $\sin 3\theta \sin \theta = \frac{1}{2}(\cos 2\theta - \cos 4\theta)$

26. $\sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x)$

27. $\dfrac{2 \cos 3x}{\sin 4x - \sin 2x} = \csc x$

In Exercises 28–31, use the information to solve the triangle shown at the right.

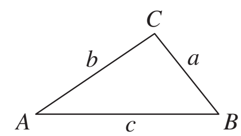


Figure for 28–31

28. $A = 46°$, $a = 14$, $b = 5$

29. $A = 32°$, $b = 8$, $c = 10$

30. $A = 24°$, $C = 101°$, $a = 10$

31. $a = 24$, $b = 30$, $c = 47$

32. Two sides of a triangle have lengths 14 inches and 19 inches. Their included angle measures 82°. Find the area of the triangle.

33. Find the area of a triangle with sides of lengths 12 inches, 16 inches, and 18 inches.

34. Write the vector $\mathbf{u} = \langle 3, 5 \rangle$ as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

35. Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$.

36. Find $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$.

37. Find k so that $\mathbf{u} = \mathbf{i} + 2k\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ are orthogonal.

38. Find the projection of $\mathbf{u} = \langle 8, -2 \rangle$ onto $\mathbf{v} = \langle 1, 5 \rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors.

39. Find the trigonometric form of the complex number shown at the right.

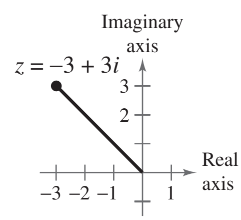


Figure for 39

40. Write the complex number $8\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$ in standard form.

41. Find the product $[4(\cos 30° + i \sin 30°)][6(\cos 120° + i \sin 120°)]$. Write the answer in standard form.

42. Find the square roots of $2 + i$.

43. Find the three cube roots of 1.

44. Write all the solutions of the equation $x^5 + 243 = 0$.

45. From a point 200 feet from a flagpole, the angles of elevation to the bottom and top of the flag are $16° 45'$ and 18°, respectively. Approximate the height of the flag to the nearest foot.

46. Write a model for a particle in simple harmonic motion with a displacement of 4 inches and a period of 8 seconds.

47. An airplane's velocity with respect to the air is 500 kilometers per hour, with a bearing of 30°. The wind at the altitude of the plane has a velocity of 50 kilometers per hour with a bearing of N 60° E. What is the true direction of the plane, and what is its speed relative to the ground?

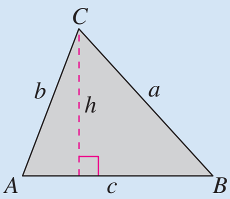
48. Forces of 60 pounds and 100 pounds have a resultant force of 125 pounds. Find the angle between the two forces.

Proofs in Mathematics

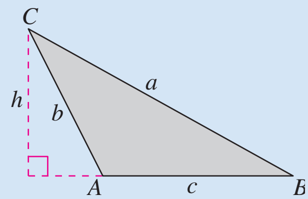
Law of Sines (p. 408)

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

Proof

Let h be the altitude of either triangle shown in the figure above. Then you have

$$\sin A = \frac{h}{b} \quad \text{or} \quad h = b \sin A$$

$$\sin B = \frac{h}{a} \quad \text{or} \quad h = a \sin B.$$

Equating these two values of h, you have

$$a \sin B = b \sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}.$$

Note that $\sin A \neq 0$ and $\sin B \neq 0$ because no angle of a triangle can have a measure of 0° or 180°. In a similar manner, construct an altitude from vertex B to side AC (extended in the obtuse triangle), as shown at the right. Then you have

$$\sin A = \frac{h}{c} \quad \text{or} \quad h = c \sin A$$

$$\sin C = \frac{h}{a} \quad \text{or} \quad h = a \sin C.$$

Equating these two values of h, you have

$$a \sin C = c \sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{c}{\sin C}.$$

By the Transitive Property of Equality, you know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

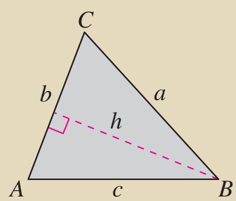
So, the Law of Sines is established.

Law of Tangents

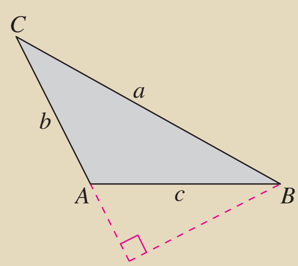
Besides the Law of Sines and the Law of Cosines, there is also a Law of Tangents, which was developed by Francois Viète (1540–1603). The Law of Tangents follows from the Law of Sines and the sum-to-product formulas for sine and is defined as follows.

$$\frac{a+b}{a-b} = \frac{\tan[(A+B)/2]}{\tan[(A-B)/2]}$$

The Law of Tangents can be used to solve a triangle when two sides and the included angle are given (SAS). Before calculators were invented, the Law of Tangents was used to solve the SAS case instead of the Law of Cosines, because computation with a table of tangent values was easier.



A is acute.



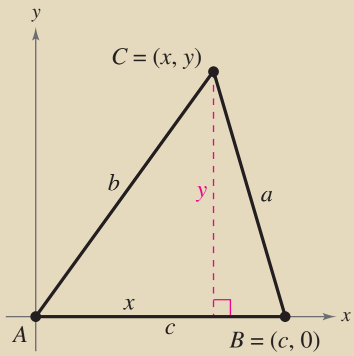
A is obtuse.

Law of Cosines (p. 417)

Standard Form	*Alternative Form*
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Proof

To prove the first formula, consider the top triangle at the right, which has three acute angles. Note that vertex B has coordinates $(c, 0)$. Furthermore, C has coordinates (x, y), where $x = b\cos A$ and $y = b\sin A$. Because a is the distance from vertex C to vertex B, it follows that



$$a = \sqrt{(x - c)^2 + (y - 0)^2} \qquad \text{Distance Formula}$$

$$a^2 = (x - c)^2 + (y - 0)^2 \qquad \text{Square each side.}$$

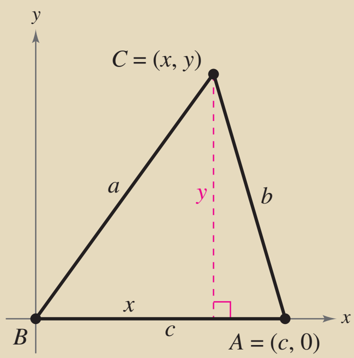
$$a^2 = (b\cos A - c)^2 + (b\sin A)^2 \qquad \text{Substitute for } x \text{ and } y.$$

$$a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \qquad \text{Expand.}$$

$$a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A \qquad \text{Factor out } b^2.$$

$$a^2 = b^2 + c^2 - 2bc\cos A. \qquad \sin^2 A + \cos^2 A = 1$$

To prove the second formula, consider the bottom triangle at the right, which also has three acute angles. Note that vertex A has coordinates $(c, 0)$. Furthermore, C has coordinates (x, y), where $x = a\cos B$ and $y = a\sin B$. Because b is the distance from vertex C to vertex A, it follows that



$$b = \sqrt{(x - c)^2 + (y - 0)^2} \qquad \text{Distance Formula}$$

$$b^2 = (x - c)^2 + (y - 0)^2 \qquad \text{Square each side.}$$

$$b^2 = (a\cos B - c)^2 + (a\sin B)^2 \qquad \text{Substitute for } x \text{ and } y.$$

$$b^2 = a^2\cos^2 B - 2ac\cos B + c^2 + a^2\sin^2 B \qquad \text{Expand.}$$

$$b^2 = a^2(\sin^2 B + \cos^2 B) + c^2 - 2ac\cos B \qquad \text{Factor out } a^2.$$

$$b^2 = a^2 + c^2 - 2ac\cos B. \qquad \sin^2 B + \cos^2 B = 1$$

A similar argument is used to establish the third formula.

Heron's Area Formula (p. 420)

Given any triangle with sides of lengths a, b, and c, the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$.

Proof

From Section 6.1, you know that

$$\text{Area} = \frac{1}{2}bc \sin A \qquad \text{Formula for the area of an oblique triangle}$$

$$(\text{Area})^2 = \frac{1}{4}b^2c^2 \sin^2 A \qquad \text{Square each side.}$$

$$\text{Area} = \sqrt{\frac{1}{4}b^2c^2 \sin^2 A} \qquad \text{Take the square root of each side.}$$

$$= \sqrt{\frac{1}{4}b^2c^2(1 - \cos^2 A)} \qquad \text{Pythagorean Identity}$$

$$= \sqrt{\left[\frac{1}{2}bc(1 + \cos A)\right]\left[\frac{1}{2}bc(1 - \cos A)\right]}. \qquad \text{Factor.}$$

Using the Law of Cosines, you can show that

$$\frac{1}{2}bc(1 + \cos A) = \frac{a+b+c}{2} \cdot \frac{-a+b+c}{2}$$

and

$$\frac{1}{2}bc(1 - \cos A) = \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}.$$

Letting $s = (a+b+c)/2$, these two equations can be rewritten as

$$\frac{1}{2}bc(1 + \cos A) = s(s-a)$$

and

$$\frac{1}{2}bc(1 - \cos A) = (s-b)(s-c).$$

By substituting into the last formula for area, you can conclude that

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Properties of the Dot Product (p. 438)

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Proof

Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, $\mathbf{w} = \langle w_1, w_2 \rangle$, $\mathbf{0} = \langle 0, 0 \rangle$, and let c be a scalar.

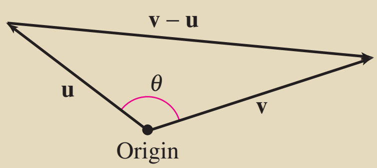
1. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0 \cdot v_1 + 0 \cdot v_2 = 0$
3. $$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \\ &= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}\end{aligned}$$
4. $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2 = \|\mathbf{v}\|^2$
5. $$\begin{aligned}c(\mathbf{u} \cdot \mathbf{v}) &= c(\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle) \\ &= c(u_1v_1 + u_2v_2) \\ &= (cu_1)v_1 + (cu_2)v_2 \\ &= \langle cu_1, cu_2 \rangle \cdot \langle v_1, v_2 \rangle \\ &= c\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

Angle Between Two Vectors (p. 439)

If θ is the angle between two nonzero vectors **u** and **v**, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

Proof

Consider the triangle determined by vectors **u, v,** and $\mathbf{v} - \mathbf{u}$, as shown in the figure. By the Law of Cosines, you can write



$$\begin{aligned}\|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.\end{aligned}$$

Progressive Summary (Chapters 1–6)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, 9 and 11. In each progressive summary, new topics encountered for the first time appear in red.

Algebraic Functions

Polynomial, Rational, Radical

Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational	Multiply by LCD
Radical	Isolate, raise to power
Absolute Value	Isolate, form two equations

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Symmetry	Transformations
Slope	Composition
Asymptotes	Standard forms of equations
End behavior	Leading Coefficient Test
Minimum values	Synthetic division
Maximum values	Descartes's Rule of Signs

Numerically

Table of values

Transcendental Functions

Exponential, Logarithmic
Trigonometric, Inverse Trigonometric

Rewriting

Exponential form ↔ Logarithmic form
Condense/expand logarithmic expressions
Simplify trigonometric expressions
Prove trigonometric identities
Use conversion formulas
Operations with vectors
Powers and roots of complex numbers

Solving

Equation	*Strategy*
Exponential	Take logarithm of each side
Logarithmic	Exponentiate each side
Trigonometric	Isolate function factor, use inverse function
Multiple angle or high powers	Use trigonometric identities

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Asymptotes	Transformations
Minimum values	Composition
Maximum values	Inverse properties
	Amplitude, period
	Reference angles

Numerically

Table of values

Other Topics

Rewriting

Solving

Analyzing